Electrical Systems

Peter Avitabile
Mechanical Engineering Department
University of Massachusetts Lowell
Electrical Systems

Passive Electrical Elements

Resistor, Inductor, Capacitor

Elemental Relations for Voltages

Resistor

\[ V_R = R \cdot i_R \]

Voltage \( V_R \)
Resistance \( R \)
Current \( i_R \)

(ohm = volt/ampere)

Resistors dissipate heat - no energy storage
**Elemental Relations for Voltages cont.**

**Inductor**

\[ \text{Voltage} \quad V_L = L \frac{di_L}{dt} \]

Inductors generate considerable resistance

\[ L = \text{Henry} = \text{rate of change of one amp/sec} \]
\[ \text{will induce an emf of one volt} \]

**Capacitor**

\[ \text{Voltage} \quad V_C = \frac{1}{C} \int i_c \, dt \]

Note \( C = q/v_c \rightarrow q\)-charge

Note also \( i = dq/dt \) and \( v_c = q/c \)

\( \text{Farad} = \frac{\text{amp sec}}{\text{volt}} \)
Elemental Relations for Currents

**Resistor**
\[ i_R = \frac{V_R}{R} \]

**Inductor**
\[ i_L = \frac{1}{L} \int V_L \, dt \]

**Capacitor**
\[ i_C = c \frac{dV_c}{dt} \]
**Electrical Systems**

*Ohm’s Law* states that the current in a circuit is proportional to the total electromotive (emf) force acting in the circuit and inversely proportional to the total resistance in the circuit.

\[
i = \frac{e}{R}
\]

\[e \approx V\]

*i* - current (amps); *e* - emf (volt); *R* - resistance (Ohm)

**Series**

\[i = \frac{e}{R} = R_{\text{effective}}\]

\[e = e_1 + e_2 + e_3\]
**Electrical Resistance**

**Parallel**

\[ e \quad \begin{array}{c} \text{R}_1 \\ \text{i}_1 \end{array} \quad \begin{array}{c} \text{R}_2 \\ \text{i}_2 \end{array} \quad \begin{array}{c} \text{R}_3 \\ \text{i}_3 \end{array} \quad e \]

\[ i_1 = \frac{e}{R_1}; \quad i_2 = \frac{e}{R_2}; \quad i_3 = \frac{e}{R_3} \]

Since

\[ i = i_1 + i_2 + i_3 \]

Then

\[ \frac{1}{R_{\text{eff}}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \]

\[ R_{\text{eff}} = \frac{R_1R_2R_3}{R_1R_2 + R_2R_3 + R_3R_1} \]
**Kirchoff’s Law**

**Two Law’s exist: Current Law (node law)**

**Voltage Law (loop law)**

**Current Law (node law)**

The algebraic sum of all currents entering and leaving a node is zero.

\[
i_1 + i_2 + i_3 = 0 \\ - (i_1 + i_2 + i_3) = 0 \\ i_1 + i_2 - i_3 = 0
\]
Kirchoff's Law

Consider the circuit

\[ V = V_L + V_R \]

\[ L \frac{di}{dt} + Ri = V \]

The instant the switch is closed, the current \( i(0) = 0 \) because the inductor cannot change from zero to a finite value instantaneously.
**Kirchoff’s Law**

**Take the Laplace Transform**

\[
L [sI(s) - i(0)] + RI(s) = \frac{E}{S}
\]

**Since** \( i(0) = 0 \)

\[
(LS + R)I(s) = \frac{E}{S}
\]

**OR**

\[
I(s) = \frac{E}{S(LS + R)} = \frac{E}{R} \left[ \frac{1}{S} - \frac{1}{S + \left( \frac{R}{L} \right)} \right]
\]

Note that \( E \) and \( V \) are often used for voltage.
Kirchoff’s Law

The inverse Laplace gives

\[ i(t) = \frac{E}{R} \left[ 1 - e^{-\left(\frac{R}{L}\right)t} \right] \]

Prove that the slope of the function intersects the final value

\[ \frac{E}{R} \text{ at } \tau = \frac{L}{R} \]
Voltage Law (Loop Law)

The algebraic sum of the voltages around the loop in an electrical circuit is zero.

\[ V_L + V_R + V_C - V = 0 \]
Voltage Law (Loop Law) cont.

Continuing with this loop, the diff. eq. is

\[ V_L = L \frac{di_L}{dt}; \quad V_R = R i_R; \quad V_C = \frac{1}{C} \int i_C dt \]

Then

\[ L \frac{di}{dt} + Ri + \frac{1}{C} \int_0^t i \ dt = V_a \]

Taking a derivative to eliminate the integral

\[ L \frac{d^2i}{dt^2} + R \frac{di}{dt} + \frac{1}{C} i = \frac{dV_a}{dt} \]

OR

\[ \frac{d^2i}{dt^2} + \frac{R}{L} \frac{di}{dt} + \frac{1}{LC} i = \frac{1}{L} \frac{dV_a}{dt} \]

Thus

\[ \omega_n = \frac{1}{\sqrt{LC}}; \quad \zeta = \frac{R}{L \omega_n^2} = \frac{1}{2} \sqrt{\frac{R^2}{L}} \]
Voltage Law (Loop Law) cont.

Taking the Laplace Transform (with IC=0)

\[(S^2 + \frac{R}{L}S + \frac{1}{LC})I(S) = \frac{S}{L}V_a(S)\]

The Transfer Function is

\[\frac{I(S)}{V(S)} = \frac{\frac{S}{L}}{S^2 + (\frac{R}{L})S + (\frac{1}{LC})}\]

Rearranging

\[(LCS^2 + RCS + 1) I(S) = CS V_a(S)\]

Recalling the relation

\[V_c = \frac{1}{c} \int i_c dt\]
Voltage Law (Loop Law) cont.

Recalling the relation

\[ V_c = \frac{1}{c} \int i_c dt \]

The Transfer Function relating the input voltage to the voltage drop across the capacitor is

\[
\frac{V_c(S)}{V_a(S)} = \frac{1}{CS} \frac{I(S)}{V_a(S)} = \frac{1}{LC} \left[ S^2 + \frac{R}{L}S + \frac{1}{(LC)} \right]
\]
Operational Amplifier

An OP-AMP is an electronic amplifier with a single output and very high voltage gain (10^5 to 10^6) - low current!

\[ e_+ \quad + \quad e_- \quad - \quad e_o = K(e_+ - e_-) \]

In General

The OP-AMP amplifies the difference in voltages \( e_+ \) and \( e_- \) and is often referred to as a differential amplifier

In this configuration, an OP-AMP is inherently unstable - to stabilize it a negative feedback is often used.
An OP-AMP is the most widely used analog electronic device.

Definitions:

- \( e_A, e_B \) ⇒ Input Voltages
- \( i_A, i_B \) ⇒ Bias Currents
- \( Z_D \) ⇒ Differential Input Impedance
- \( A \) ⇒ Open Loop Gain
- \( Z_o \) ⇒ Output Impedance
- \( V_{os} \) ⇒ Offset Voltage
- \( \pm V_s \) ⇒ Voltage Supply

\[
e_o = (e_A - e_B - V_{os}) A
\]
Operational Amplifier

Voltage Follower

\[ e_o = (e_A - e_B - V_{os})A \]

Assume \( A \to \infty \) and \( V_{os} \to 0 \)

\[ \frac{e_o}{\infty} = e_i - e_o \quad : \quad e_o = e_i \]

Output follows input

Source supplying \( e_i \) works into an “infinite” impedance thus no current is drawn

\( e_o \) does not draw current or consume power

Sometimes called a buffer amplifier
**Operational Amplifier**

*Consider an inverting amplifier shown*

![Operational Amplifier Circuit Diagram]

*Since only negligible current flows into the amplifier, \( i_1 \) must equal \( i_2 \)*

\[
\begin{align*}
    i_1 &= \frac{(e_i - e')}{R_1} \\
    i_2 &= \frac{(e' - e_o)}{R_2}
\end{align*}
\]

*Since \( K >> 1 \) then \( e' \approx 0 \). Thus the approximate model is:*

\[
\begin{align*}
    \frac{e_i}{R_1} &= -\frac{e_o}{R_2} \\
    e_o &= -\frac{R_2}{R_1} e_i \\
    \text{Gain is } &\Rightarrow \frac{R_2}{R_1}
\end{align*}
\]
**Operational Amplifier**

**Consider a non-inverting amplifier shown**

![Operational Amplifier Diagram]

**For this circuit**

\[ e_0 = K(e_i - \frac{R_1}{R_1 + R_2} e_0) \]

or

\[ e_i = \left( \frac{R_1}{R_1 + R_2} + \frac{1}{K} \right) e_0 \]

**Since** \( K \gg 1 \)

**And if** \( \frac{R_1}{R_1 + R_2} \gg \frac{1}{K} \)

**Then an approximate model is**

\[ e_o = \left( \frac{R_2}{R_1} + 1 \right) e_i \]

Gain is \( \Rightarrow \frac{R_2}{R_1} \)
Low-Pass Filter

Transfer Function

\[ \frac{e_o(s)}{e_i} = \frac{X_o(s)}{X_i} \]

\[ = \frac{1}{\tau s + 1} \]

Frequency response function is evaluated at

\[ s = j\omega \]

Electrical

\[ \frac{e_i}{R} \quad \frac{e_o}{C} \quad \tau \approx RC \]

Mechanical

\[ \frac{X_i}{K} \quad \frac{X_o}{B} \quad \tau \approx \frac{B}{K} \]
Low-Pass Filter

Passive styled filters have various advantages:

Very low noise output

No power required

Versatile

Wide dynamic range
High-Pass Filter

Transfer Function

\[ \frac{e_o(s)}{e_i} = \frac{X_o(s)}{X_i} = \frac{\tau s}{\tau s + 1} \]

Frequency response function is evaluated at

\[ s = j\omega \]

Electrical

Mechanical
**Band-Pass Filter**

**Cascade a low pass and high pass filter together**

\[
\frac{1}{\tau_1 s + 1} \quad \frac{\tau_2 s}{\tau_2 s + 1}
\]

\[\tau_2 > \tau_1\]
Integration (Approximation Using Low Pass)

\[
\frac{e_o(j\omega)}{e_i} = \frac{1}{j\omega \tau + 1}
\]

Now if \( \omega \tau \gg 1 \) then

\[
\frac{e_o(j\omega)}{e_i} = \frac{1}{j\omega \tau}
\]

Or

\[
\frac{e_o(s)}{e_i} = \frac{1}{\tau s} \quad \therefore \quad e_o \approx \frac{1}{\tau} \int e_i \, dt
\]
Differentiation (Approximation Using High Pass)

\[ \frac{e_o(j\omega)}{e_i} = \frac{j\omega \tau}{j\omega \tau + 1} \]

Now if \( \omega \tau << 1 \) then

\[ \frac{e_o(j\omega)}{e_i} = j\omega \tau \]

Or

\[ \frac{e_o(s)}{e_i} \approx \tau s \quad \therefore \quad e_o = \tau \frac{de_i}{dt} \]

BEWARE!!
These circuits are sensitive to noise upon differentiating
Differentiation (w/Low Pass Filter)

\[ \frac{e_o(s)}{e_i} = -\frac{R_2Cs}{R_1Cs + 1} \]

Accurate for \( R_1C\omega \ll 1 \)

But amplifies high frequency noise by \( \frac{R_2}{R_1} \)
Differentiation (w/Low Pass Filter & Noise Atten.)

\[
\frac{e_o(s)}{e_i} = -\frac{R_2C_1s}{(R_2C_2s + 1)(R_1C_1s + 1)}
\]
**ElectroMechanical Systems**

Armature-controlled DC motors are popular. Field Controlled DC motors are not as popular.

**Elemental Relations**

Electrical and mechanical parts are coupled. The mechanical motion of the rotor relative to the stator creates an electromotive force voltage (EMF effect). The back EMF voltage across the DC motor is proportional to angular speed of the rotor.

\[ e_b = K_e \omega = K_e \dot{\theta} \]

while the torque developed by the motor is proportional to the current

\[ T = K_T i \]
**Electromechanical Systems**

\[ T = K_T i \]

*where*

- \( e_b \) – back emf voltage
- \( \theta \) – angular displacement
- \( \dot{\theta} = \omega \) – angular velocity
- \( T \) – torque applied to rotor (exerted by motor)
- \( K_e \) – emf constant of motor \( \left( \frac{V}{K_{\text{RPM}}} \right) \)
- \( K_t \) – torque constant of motor \( \left( \frac{\text{oz}_f}{\text{Amp}} \right) \)

*Conversion:*

\[
1 \frac{\text{in} - \text{oz}_f}{\text{A}} = 0.741 \frac{V}{K_{\text{RPM}}}
\]
Armature-Controlled DC Motors

\[ T_L(t) = \frac{1}{2} k c^2 \theta + c d \dot{\theta} + J_d \ddot{\theta} \]

- \( c \) – damping of shaft
- \( c_d \) – damping of disk
- \( i_a \) – armature current
- \( i_f \) – field current (constant)
- \( J_d \) – mass inertia of disk
- \( J_R \) – mass inertia of rotor
- \( k \) – torsional stiffness
- \( L_a \) – armature inductance
- \( R_a \) – armature resistance
- \( v_a \) – armature voltage
Armature-Controlled DC Motors

Rigid Shaft - Lump the rotor, shaft and disk together so that $J = J_r + J_d$

One input, $v_a$

Two outputs, $i_a$ and $\theta$

For the electrical circuit, the loop method gives

$$\sum V_{\text{drop}} = 0 \quad v_R + v_L + e_b - v_a = 0$$

Then,

$$R_a i_a + L_a \frac{di_a}{dt} + e_b = v_a \quad (e_b = K_e \omega = K_e \dot{\theta})$$

So that

$$L_a \frac{di_a}{dt} + R_a i_a + K_e \dot{\theta} = v_a$$
Armature-Controlled DC Motors

For the mechanical part

\[ T + T_L - B_d \dot{\theta} = J \ddot{\theta} \]

Then

\[ J \ddot{\theta} + B_d \dot{\theta} - K_t i_a = T_L \]

(Note: Voltage is input to system but current is input to mechanical part)

System Equations

\[ J \ddot{\theta} + B_d \dot{\theta} - K_t i_a = T_L \]

\[ L_a \frac{d}{dt} i_a + R_a i_a + K_e \dot{\theta} = v_a \]

OR

\[
\begin{bmatrix}
J & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\ddot{\theta} \\
\frac{d^2 i_a}{dt^2}
\end{bmatrix}
+ \begin{bmatrix}
B_d & 0 \\
K_e & L_a
\end{bmatrix}
\begin{bmatrix}
\dot{\theta} \\
\frac{di_a}{dt}
\end{bmatrix}
+ \begin{bmatrix}
0 & -K_t \\
0 & R_a
\end{bmatrix}
\begin{bmatrix}
\theta \\
i_a
\end{bmatrix}
= \begin{bmatrix}
T_L \\
v_a
\end{bmatrix}
\]
Armature-Controlled DC Motors

Expressing the equations using $\omega$ (since $\omega = \dot{\theta}$)

\[ J \dot{\omega} + B_d \omega - K_t i_a = T_L \]

\[ L_a \frac{d i_a}{dt} + R_a i_a + K_e \omega = v_a \]

\[
\begin{bmatrix}
    J & 0 \\
    0 & L_a
\end{bmatrix}
\begin{bmatrix}
    \dot{\omega} \\
    \frac{d i_a}{dt}
\end{bmatrix}
+
\begin{bmatrix}
    B_d & -K_t \\
    K_e & R_a
\end{bmatrix}
\begin{bmatrix}
    \omega \\
    i_a
\end{bmatrix}
=
\begin{bmatrix}
    T_L \\
    v_a
\end{bmatrix}
\]
Armature-Controlled DC Motors - Laplace Transform

The Laplace Transform gives

\[
\begin{bmatrix}
Js + B_d & -K_t \\
K_e & La s + R_a
\end{bmatrix}
\begin{bmatrix}
\Omega(s) \\
I_a(s)
\end{bmatrix}
= \begin{bmatrix}
T_L(s) \\
V_a(s)
\end{bmatrix}
\]

After some manipulation, the characteristic equation is given by

\[
JL_a s^2 + \left( B_d L_a + JR_a \right)s + B_d R_a + K_e K_t = 0
\]

After putting this in standard form,

\[
\omega_n^2 = \frac{(B_d R_a + K_e K_t)}{JL_a}
\]

\[
2\zeta\omega_n = \frac{(B_d L_a + JR_a)}{JL_a}
\]
The transfer functions of interest are

$$\frac{\Omega(s)}{V_a(s)} = \frac{K_t}{JL_a s^2 + (B_d L_a + J R_a) s + B_d R_a + K_e K_t}$$

$$\frac{\theta(s)}{V_a(s)} = \frac{K_t}{JL_a} \left[ \frac{1}{s \left[ s^2 + \frac{B_d L_a + J R_a}{JL_a} s + \frac{B_d R_a + K_e K_t}{JL_a} \right]} \right]$$

Note: A more complex model which includes a flexible shaft can also be developed.
Example - Circuit Attenuation

A low frequency signal (at 3 Hz) is to be measured but is contaminated by a higher frequency component such as 60 Hz noise for instance.

A first order low pass filter (RC circuit)

The filter must be selected to attenuate the 60 Hz noise without significantly affecting the actual signal attenuation by no more than a factor of 0.9.
Example – Circuit Attenuation

Filter Transfer Function

\[ H(s) = \frac{V_{\text{out}}(s)}{V_{\text{in}}(s)} = \frac{1}{\text{RC}s + 1} \]

The magnitude of the frequency response function is

\[ |H(j\omega)| = \left| \frac{V_{\text{out}}}{V_{\text{in}}} \right| = \frac{1}{\sqrt{(\omega \tau)^2 + 1}} = 0.9 \]

\[ \tau = \text{RC time constant} = 0.026 \text{ s} \]

Since \[ 0.9 = \frac{1}{\sqrt{\tau \cdot 2\pi(3\text{Hz})^2 + 1}} \]

At 60 Hz, the amplitude is \[ \left[ (0.0257 \cdot 2 \cdot 60 \cdot \pi)^2 + 1 \right]^{-\frac{1}{2}} = 0.093 \]

with RC = 0.026s – possible values \[ \{ \begin{align*} C &= 0.2\mu\text{F} \\ R &= 130\text{k}\Omega \end{align*} \]
# Force Voltage Analogy

<table>
<thead>
<tr>
<th>Mechanical</th>
<th>Electrical</th>
</tr>
</thead>
<tbody>
<tr>
<td>Force</td>
<td>$f$</td>
</tr>
<tr>
<td>Mass</td>
<td>$m$</td>
</tr>
<tr>
<td>Viscous Friction</td>
<td>$c$</td>
</tr>
<tr>
<td>Spring Constant</td>
<td>$k$</td>
</tr>
<tr>
<td>Displacement</td>
<td>$x$</td>
</tr>
<tr>
<td>Velocity</td>
<td>$\dot{x}$</td>
</tr>
</tbody>
</table>

| Voltage          | $v$                         |
| Resistance       | $L$                         |
| Inductance       | $R$                         |
| Reciprocal of Capacitance | $\frac{1}{C}$          |
| Charge           | $q$                         |
| Current          | $i$                         |

$$ m\ddot{x} + c\dot{x} + kx = f(t) \quad \quad L\dot{q} + Rq + \frac{1}{C}q = v $$
Force Current Analogy

<table>
<thead>
<tr>
<th>Mechanical</th>
<th>Electrical</th>
</tr>
</thead>
<tbody>
<tr>
<td>Force</td>
<td>$f$</td>
</tr>
<tr>
<td>Mass</td>
<td>$m$</td>
</tr>
<tr>
<td>Viscous Friction</td>
<td>$c$</td>
</tr>
<tr>
<td>Spring Constant</td>
<td>$k$</td>
</tr>
<tr>
<td>Displacement</td>
<td>$x$</td>
</tr>
<tr>
<td>Velocity</td>
<td>$\dot{x}$</td>
</tr>
</tbody>
</table>

$$m\ddot{x} + c\dot{x} + kx = f(t)$$

$$C\dot{\psi} + \frac{1}{R}\psi + \frac{1}{L}\psi = i$$