Complex Analysis, Differential Equations, and Laplace Transform

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Complex Analysis

\[ z = x + jy = x + iy \]

*Where:* \[ i = j = \sqrt{-1} \]

*Real part of the function* \[ \text{Re}(z) = x \]

*Imaginary part of the function* \[ \text{Im}(z) = y \]

*Addition*

\[
\begin{align*}
    z_1 &= x_1 + jy_1 \\
    z_2 &= x_2 + jy_2
\end{align*}
\]

\[ z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2) \]
Complex Numbers - Multiplication and Division

**Multiplication**

\[ z_1 \cdot z_2 = (x_1 x_2 - y_1 y_2) + j(x_1 y_2 + x_2 y_1) \]

\[ z_1 \cdot z_1^* = x_1^2 + y_1^2 \]

where: \( z_1^* = \) complex conjugate

and: \( z_1^* = x - jy \)

if: \( z_1 = x + jy \)

**Division**

\[
\frac{x_1 + jy_1}{x_2 + jy_2} = \frac{x_1 + jy_1}{x_2 + jy_2} \cdot \frac{x_2 - jy_2}{x_2 - jy_2}
\]

\[
= \frac{(x_1 x_2 + y_1 y_2) + j(y_1 x_2 - y_2 x_1)}{x_2^2 + y_2^2}
\]
Example - Complex Number Multiplication

Perform the following multiplication and express the result in rectangular form.

\((-2 + j5)(3 - j2)\)

\[ \text{Solution: Treating the two complex numbers as binomials, the product is obtained as} \]

\[ (-2 + j5)(3 - j2) = -6 + j4 + j15 - j^210 \]

\[ = -6 + j19 + 10 \]

\[ = 4 + j19 \]
Example – Complex Number Division

Perform the following division of complex numbers and express the result in rectangular form.

\[
\frac{-1 + j3}{2 + j5} + \frac{5j2}{3j1} - \frac{29}{13}j13
\]

Solution:

\[
\frac{-1 + j3}{2 + j5} \cdot \frac{2 - j5}{2 - j5} = \frac{13 + j11}{29} = \frac{13}{29} + \frac{j11}{29}
\]
Example - Complex Conjugate

Given: \( z = -1 + 2j \)  Evaluate: \( zz^* \)

Solution:

\[
zz^* = (-1 + 2j)(-1 - 2j) = (-1)^2 + (2)^2 = 5
\]
Complex Conjugates

Location of Complex Conjugates in the Complex Plane

\[-x \pm jy\]
Complex Numbers in Polar Form

\[ z = x + jy \]

\[ x = r \cos \theta \]

\[ y = r \sin \theta \]
Polar Form - Euler’s Formula

\[ e^{j\theta} = \cos \theta + j \sin \theta \]
\[ z = x + jy = r \cos \theta + jr \sin \theta \]
\[ z = r(\cos \theta + j \sin \theta) = r e^{j\theta} \]

\[ r = |z| = \sqrt{\text{Re}\{z\}^2 + \text{Im}\{z\}^2} = \sqrt{x^2 + y^2} \]
\[ \theta = \tan^{-1}\left(\frac{y}{x}\right) \quad (\theta \text{ positive in ccw direction}) \]
Example

Express the complex number

\[ z = 3 + j\sqrt{3} \]

in polar form.

Solution:

\[ |z| = \sqrt{12} = 2\sqrt{3} \quad \theta = \tan^{-1}\left(\frac{\sqrt{3}}{3}\right) = \frac{\pi}{6} \]

\[ z = 2\sqrt{3}e^{j\left(\frac{\pi}{6}\right)} \]
Example

Determine the location and the phase angle of the complex number:

\[
\frac{2}{-1+j}
\]

Solution: Express this number in standard rectangular form by multiplying its numerator and denominator by the conjugate of the denominator.

\[
\frac{2}{-1+j} \cdot \frac{-1-j}{-1-j} = \frac{-2-2j}{1+1} = -1-j
\]

Location = third quadrant

Phase Angle = +225° or -135°
Multiplication and Division (Polar Form)

Multiplication
\[ z_1 = r_1 e^{j\theta_1}; \quad z_2 = r_2 e^{j\theta_2} \]
\[ z_1 z_2 = r_1 r_2 e^{j(\theta_1 + \theta_2)} \]

Division
\[ \frac{z_1}{z_2} = \frac{r_1}{r_2} e^{j(\theta_1 - \theta_2)} \]

Complex Conjugate
\[ z_1^* = re^{-j\theta_1} \]
\[ z \cdot z^* = re^{j\theta_1} \cdot re^{-j\theta_1} = r^2 \]
Complex Functions

Complex Variable, $s$

$s = \sigma + j\omega$

\[ G(s) = K \frac{s^m + a_1s^{m-1} + \ldots + a_{m-1}s + a_m}{s^n + b_1s^{n-1} + \ldots + b_{n-1}s + b_n} \]

\[ = \frac{N(s)}{D(s)} \] (where $m < n$)
Complex Function

The complex function can be expressed in POLE-ZERO form as:

\[ G(s) = K \frac{(s-z_1)(s-z_2)\ldots(s-z_m)}{(s-p_1)(s-p_2)\ldots(s-p_n)} \]

The roots of the numerator are referred to as ZEROS. The roots of the denominator are referred to as POLES.

Often this can be written in partial fraction form as:

\[ G(s) = \frac{a_1}{s-p_1} + \frac{a_2}{s-p_2} + \ldots + \frac{a_n}{s-p_n} \]
Example

Express the given complex function in pole-zero form. Identify the zeros and the poles, as well as the multiplicity of each.

\[
G(s) = \frac{2s + 1}{s(s + 2)^2(10s + 3)}
\]

Solution: \( G(s) \) can be written in pole-zero form as:

\[
G(s) = \frac{2(s + 0.5)}{10s(s + 2)^2(s + 0.3)}
\]

\[
= \left( \frac{1}{5} \right) \frac{s + 0.5}{s(s + 2)^2(s + 0.3)}
\]

Simple zero: \( s = -0.5 \)

Two simple poles: \( s = 0 \) and \( s = -0.3 \)

Double pole: \( s = -2 \)
Differential Equations

1st Order ODE

\[ \frac{1}{\tau} \ddot{x} + x = f(t) \quad (\tau > 0) \]

\( \tau \) is the time constant

2nd Order ODE

\[ \ddot{x} + 2\zeta \omega_n \dot{x} + \omega_n^2 x = f(t) \]

\( \zeta \) - damping ratio

\( c_c \) - critical damping

\( \omega_n \) - natural frequency

\[ \zeta = \frac{c}{c_c}; \quad c_c = 2m\omega_n; \quad \omega_n^2 = \frac{k}{m} \]
2nd Order ODE

2nd order - 2 integrals/derivatives - 2 constants

Initial Value Problem

Constant Coefficients - time independent

x is dependent variable - t is independent

If coefficients do not depend on x, then the equations are linear (linear superposition possible)

Has $X_h$ and $X_p$ - homogeneous & particular
### Example - Types of ODEs

<table>
<thead>
<tr>
<th>Order</th>
<th>Description</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>First-order</td>
<td>linear with constant coefficients</td>
<td>(2\dot{x} + x = 0)</td>
</tr>
<tr>
<td>Second-order</td>
<td>linear with constant coefficients</td>
<td>(\ddot{x} + 3\dot{x} + 9x = 2\sin t)</td>
</tr>
<tr>
<td>Second-order</td>
<td>linear</td>
<td>(\ddot{x} + (2t-1)\dot{x} + 2x = 0)</td>
</tr>
<tr>
<td>First-order</td>
<td>nonlinear</td>
<td>(\dot{x} + (\sin t)x = \sin 3t)</td>
</tr>
<tr>
<td>First-order</td>
<td>nonlinear</td>
<td>(2\dot{x} + x^2 = t)</td>
</tr>
<tr>
<td>Second-order</td>
<td>nonlinear</td>
<td>(\ddot{x} + (x+1)\dot{x} + 9x = 0)</td>
</tr>
</tbody>
</table>

If the coefficients are constants or functions of \(t\), the ODE is linear. Otherwise it is non-linear.

*Highest derivative identifies the order*
Example – 1st Order ODE

Consider the single-tank, liquid-level system shown in the figure below. The mathematical model of this system is given by the following first-order, linear ODE with constant coefficients.

\[
\frac{RA}{g} \dot{h} + h = \frac{R}{g} q_i(t)
\]
Example – 1st Order ODE

The ODE can easily be expressed in the standard form as:

\[ \dot{h} + \frac{g}{RA} h = \frac{1}{A} q_i(t) \]

As a result, the system’s time constant is identified as:

\[ \tau = \frac{RA}{g} \]
Free Response - First Order

Homogeneous Solution

\[
\dot{x} + \frac{1}{\tau} x = 0 \quad \tau > 0
\]

Characteristic Equation

\[
\lambda + \frac{1}{\tau} = 0 \quad \Rightarrow \lambda = -\frac{1}{\tau}
\]

The solution becomes

\[
x(t) = ce^{-t/\tau}
\]
Step Response - First Order

Consider a 1\textsuperscript{st} order system described as:

\[ A\dot{x} + Bx = f(t) \]

And subjected to a step input.

\[ \tau \frac{dx}{dt} + x = \frac{f(t)}{B} \]

In standard form:

\[ \frac{dx}{dt} + \frac{1}{\tau} x = \frac{f(t)}{A} \]
**Step Response - First Order**

**First Order homogeneous response is in the form of an exponential function**

\[ y_h(t) = e^{-\lambda t} \quad \text{where} \quad \lambda = \frac{1}{\tau} \]

The solution is

\[ y(t) = Ce^{-\frac{t}{\tau}} + y_p(t) \]

where \( C \) is determined from initial condition for a particular solution and \( y_p \) indicates the particular solution.
Step Response - First Order

If the system is subjected to a step change

\[ f(t) = \mu_s(t) \begin{cases} 
  = 0 & t \leq 0 \\
  = 1 & t > 0 
\end{cases} \]

The particular solution can be found to be

\[ y(t) = Ce^{-t/\tau} + 1 \]

Given the initial conditions when the system is initially at rest (t=0, \( y_s(0)=0 \)) requires that C=-1 which then gives

\[ y(t) = 1 - e^{-t/\tau} \]
Step Response - First Order

Step Response - First Order

% Plotting a step response using a Matlab M-file
% Note: This code assumes that the solution to the differential equation has already been obtained
% Note: Initial conditions are assumed to be zero

% Define the length of time in increments of 0.01 seconds.
t = 0:0.01:3;

% Define the time constant | this example code utilizes an arbitrary time constant |
T = .25;

% Solution of the differential equation
y = 1-exp(-t/T);

figure (1)
pplot(t,y)
xlabel('Time (s)')
ylabel('Displacement')
title('Step Response')
Step Response - First Order

Response is broken up into two regions:

1. Transient region where system is still responding dynamically

2. Steady-state region where system has reached its final value

Note: There is no clear break point; often, $3\tau$, $4\tau$, or $5\tau$ is chosen based on desired accuracy

The initial slope of the response may be found by differentiating $y(t)$ to give

$$\frac{dy}{dt} = \frac{1}{\tau}$$
Step Response - First Order

Using the following equation

\[ A\dot{x} + Bx = f(t) \]

HOMEWORK ASSIGNMENT -----

Calculate the response by hand and plot by hand. Let \( A = \text{month} \) and \( B = \text{day of your birthday} \)

Use MATLAB to confirm your results
Transfer Function of Step Response

Diff. Eq. \[ A\dot{x} + Bx = f(t) \]

reduces to \[ \left(\frac{A}{B}\right)\dot{x} + x = \frac{1}{B}f(t) \]

or \[ \dot{x} + \left(\frac{1}{\frac{A}{B}}\right)x = \frac{1}{A}f(t) \]

Transferring to the LaPlace domain:

Transfer Function is \[ \frac{x(s)}{f(s)} = \frac{1}{A} \frac{1}{s + \frac{B}{A}} \]
Step Response - First Order

Using MATLAB and the equivalent Laplace form, the system transfer function is described as

\[
\frac{1}{A} \left( S + \frac{B}{A} \right)^{-1} = \frac{1}{A} \frac{1}{\left( S + \frac{B}{A} \right)}
\]

\[\text{NUM} = [0 \ 1/A];\]
\[\text{DEN} = [1 \ B/A];\]
\[\text{step}(\text{NUM}, \text{DEN})\]
Free Response - Second Order

Homogeneous Solution

\[ \ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = 0 \]

Characteristic Solution

\[ \lambda^2 + 2\zeta\omega_n \lambda + \omega_n^2 = 0 \]

\[ \lambda_{1,2} = -\zeta\omega_n \pm \sqrt{(\zeta\omega_n)^2 - \omega_n^2} \]

\[ = -\zeta\omega_n \pm \omega_n \sqrt{\zeta^2 - 1} \]

\[ \zeta = \frac{c}{c_c}; \quad c_c = 2 m \omega_n; \quad \omega_n^2 = \frac{k}{m} \]
Free Response - Second Order

For purposes of development of these general equations a simple mass, spring, dashpot system will be used.

Equation of motion is obtained from Newton's second law

\[ m \frac{d^2x}{dt^2} + c \frac{dx}{dt} + kx = f(t) \]

With I.C.'s

\[ x(0) = x_0 \quad \text{and} \quad \dot{x}(0) = \dot{x}_0 \]
Free Response - Second Order

A solution of the form \( x = e^{\lambda t} \) fits

\[
(m\lambda^2 + c\lambda + k)e^{\lambda t} = 0
\]

Characteristic equation

\[
m\lambda^2 + c\lambda + k = 0 \Rightarrow \lambda = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}
\]

Solution has 2 roots and 3 possible solutions depending on the term under the \( \sqrt{\ } \).
Free Response - Case 1 \[ c^2 < 4mk \quad (\zeta < 1) \]

\[ \lambda = \frac{-c \pm \sqrt{-(4mk - c^2)}}{2m} \]

\[ \lambda_{1,2} = -\frac{c}{2m} \pm \frac{\sqrt{4mk - c^2}}{2m} j \]

\[ = -\alpha \pm \beta j \quad (Two \ complex \ conjugate \ roots) \]

The solution is

\[ x_h = Ae^{(-\alpha + \beta j)t} + Be^{(-\alpha - \beta j)t} \]
Free Response - Case 1

Factor out: \( e^{-\alpha t} \)

\[ \Rightarrow x_h = e^{-\alpha t} \left( A e^{j\beta t} + B e^{-j\beta t} \right) \]

Recall that:
\[ e^{j\theta} = \cos \theta + j \sin \theta; \quad e^{-j\theta} = \cos \theta - j \sin \theta \]

Then,
\[ x_h = e^{-\alpha t} \left[ A (\cos \beta t + j \sin \beta t) + B (\cos \beta t - j \sin \beta t) \right] \]
\[ = e^{-\alpha t} \left[ c_1 \cos \beta t + c_2 j \sin \beta t \right] \]
Free Response - Case 1

Using \( \sin(x + y) = \sin x \cos y + \cos x \sin y \)

equation can be written as

\[
x_h = ce^{-\alpha t} \sin(\beta t + \phi) \quad \left( \alpha = \frac{c}{2m} \right)
\]

\[
x_h = ce^{-\frac{c}{2m} t} \sin(\frac{\sqrt{4mk - c^2}}{2m} t + \phi)
\]
Free Response - Case 1

Now if we divide through by m, then

\[ \ddot{x} + 2\zeta \omega_n \dot{x} + \omega_n^2 x = \frac{f}{m} = 0 \quad \text{(homogeneous)} \]

where

\[ \zeta = \text{damping ratio} = \frac{c}{c_c} = \frac{c}{2\sqrt{km}} \]

\[ \omega_n = \text{natural frequency} = \sqrt{\frac{k}{m}} \]

\[ \Rightarrow x = e^{-\zeta \omega_n t} \sin(\omega_d t + \phi) \]

where \[ \omega_d = \omega_n \sqrt{1 - \zeta^2} \]
Free Response - Case 1

1. For small $\zeta \rightarrow \omega_n \approx \omega_d$
2. $\omega_n$ is independent of damping
3. If $c = 0$, then $\omega_n \equiv \omega_d$
4. Solution response is always of the form of a damped exponentially decaying sinusoidal form
**Free Response - Case 2**

\[ \zeta > 1 \] \[ c^2 > 4mk \]

\[ \lambda = \frac{c}{2m} \pm \frac{\sqrt{c^2 - 4mk}}{2m} \]

**Two REAL Roots**

\[ x = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \]

**Recall**

\[ e^\theta = \sinh \theta + \cosh \theta; \quad e^{-\theta} = \sinh \theta - \cosh \theta \]

**Then**

\[ x = c_3 \sinh \lambda_1 t + c_4 \cosh \lambda_2 t \]

**Solution of this type will always be of the form of a lag in the system.**
Free Response - Case 2

Response will have an exponential envelope but will not have oscillatory motion about steady-state.

Damping $\zeta > 1$
Free Response - Case 3  \[ c^2 = 4mk \quad (\zeta = 1) \]

\[ \lambda_1 = \lambda_1 = -\frac{c}{2m} \]

Two REAL REPEATED Roots

\[ x = c_1 e^{\lambda_1 t} + c_2 t e^{\lambda_2 t} \]

Solution of this type will also be in the form of a lag to the system, but this system will return to steady state faster than any other damping without overshoot.

This is the break point between structural dynamics and controls problems.
Free Response - Summary

\[ \zeta < 1 \]

\[ \zeta = 1 \]

\[ \zeta > 1 \]
**S-plane Representation**

- **X** - Conjugate \( \zeta < 1 \)
- **Poles**
- **○** - Repeated \( \zeta = 1 \)
- **Roots**
- **△** - Real Roots \( \zeta > 1 \)

\[
\lambda_1 = -\zeta \omega_n + j \omega_d \\
\lambda_{1,2} = -\zeta \omega_n \\
\lambda_1^* = -\zeta \omega_n - j \omega_d
\]
S-PLANE PLOTS FOR IMPULSE RESPONSE OF A SINGLE DEGREE OF FREEDOM MECHANICAL SYSTEM
**Example 1.12.** Consider the RLC (resistance, inductance, and capacitance) circuit in Fig. 1.8. The mathematical model of this electrical system will be derived in Chapter 5 and shown to be described by the following second-order, linear ODE with constant coefficients

![RLC Circuit Diagram]

**FIGURE 1.8** RLC circuit.
Example - 1.12 (cont)

\[ L\ddot{q} + R\dot{q} + \frac{1}{C}q = e(t) \]  \hspace{1cm} (1.23)

where \( q \) denotes the electric charge and is defined by \( i = dq/dt \). Rewrite Eq. (1.23) to resemble the standard form, Eq. (1.22), to obtain

\[ \ddot{q} + \frac{R}{L}\dot{q} + \frac{1}{LC}q = \frac{1}{L}e(t) \]  \hspace{1cm} (1.24)

Direct comparison of Eq. (1.24) with the standard form reveals

\[ \omega_n^2 = \frac{1}{LC}, \quad 2\zeta\omega_n = \frac{R}{L} \]

the solution of which identifies the (undamped) natural frequency and the damping ratio, as

\[ \omega_n = \sqrt{\frac{1}{LC}}, \quad \zeta = \frac{R}{2\sqrt{LC}} \]
**Example - 1.13**

**Example 1.13.** Solve the following homogeneous, second-order ODE: $\ddot{x} + 3\dot{x} + 2x = 0$.

**Solution.** The characteristic equation and values are obtained as

$$\lambda^2 + 3\lambda + 2 = 0 \implies (\lambda + 1)(\lambda + 2) = 0 \implies \lambda_{1,2} = -1, -2$$

Thus, the two real and distinct characteristic values correspond to $e^{-t}$ and $e^{-2t}$, which are clearly linearly independent. The general solution to the ODE is then expressed as the linear combination of these functions; i.e.,

$$x(t) = c_1 e^{-t} + c_2 e^{-2t}$$
**Example 1.14.** Find the general solution to the third-order ODE given by $\ddot{x} + 4\dot{x} + 5\dot{x} + 2x = 0$.

**Solution.** The characteristic equation and values are obtained as

$$\lambda^3 + 4\lambda^2 + 5\lambda + 2 = 0 \iff (\lambda + 2)(\lambda + 1)^2 = 0$$

$$\implies \lambda_1 = -2, \quad \lambda_{2,3} = -1$$

The first characteristic value is real and simple; hence it corresponds to $e^{-2t}$. The second one, however, has a multiplicity of 2 (i.e., $m = 2$). Therefore, it corresponds to $e^{-t}$ and $te^{-t}$. The general solution to the ODE is

$$x(t) = c_1 e^{-2t} + c_2 e^{-t} + c_3 te^{-t}$$

(corresponding to $\lambda = -1$ of multiplicity 2)
Example - 1.16

Example 1.16. Find the solution to the following initial-value problem: $\dot{\omega} + 2\omega = t, \omega(0^-) = 1$.

Solution

Homogeneous solution, $\omega_h(t)$. The characteristic equation and the corresponding characteristic value are

$$\lambda + 2 = 0 \implies \lambda = -2$$

which results in $\omega_h(t) = ce^{-2t}$.

Particular solution, $\omega_p(t)$. The function on the right-hand side, known as the \textit{forcing function}, is a first-degree polynomial and agrees with the first row of Table 1.1. Thus, the particular solution assumes the form $\omega_p(t) = At + B$. Furthermore, we notice that the forcing function does not coincide with the homogeneous solution. Insert $\omega_p(t)$ into the original differential equation to obtain
Example - 1.16 (cont)

\[ 2At + (A + 2B) = t \quad \Rightarrow \quad \begin{cases} 2A = 1 \\ A + 2B = 0 \end{cases} \quad \Rightarrow \quad A = \frac{1}{2}, \quad B = -\frac{1}{4} \]

Consequently, \( \omega_p(t) = \frac{1}{2}t - \frac{1}{4} \).

**General solution.** \( \omega(t) = \omega_h(t) + \omega_p(t) = ce^{-2t} + \frac{1}{2}t - \frac{1}{4} \).

**Initial conditions.** Apply the given initial condition (I.C.) to the general solution to obtain

\[ \left. \omega(t) \right|_{t=0} = 1 = c - \frac{1}{4} \quad \Rightarrow \quad c = \frac{5}{4} \]

As a result, the solution to this initial-value problem is \( \omega(t) = \frac{1}{4}(5e^{-2t} + 2t - 1) \).
Example - 1.17

Example 1.17. Solve the following initial-value problem: $\ddot{x} + 3\dot{x} + 2x = 2e^{-4t}$, $x(0^-) = 1$, $\dot{x}(0^-) = -1$.

Solution

Homogeneous solution, $x_h(t)$. The characteristic equation and values associated with the homogeneous equation are

$$\lambda^2 + 3\lambda + 2 = 0 \implies \lambda_{1,2} = -1, -2$$

which correspond to $x_h(t) = c_1e^{-t} + c_2e^{-2t}$.

Particular solution, $x_p(t)$. The function on the right-hand side agrees with the second row of Table 1.1; and hence $x_p(t) = ke^{-4t}$. Since this exponential function does not coincide with either of the homogeneous independent solutions, further modification is not needed. Substitute $x_p(t)$ into the original ODE to obtain

$$16ke^{-4t} - 12ke^{-4t} + 2ke^{-4t} = 2e^{-4t} \implies 6ke^{-4t} = 2e^{-4t}$$

$$\implies k = \frac{1}{3} \implies x_p(t) = \frac{1}{3}e^{-4t}$$
Example - 1.17 (cont)

General solution, $x(t)$.  

$$x(t) = x_h(t) + x_p(t) = c_1 e^{-t} + c_2 e^{-2t} + \frac{1}{3} e^{-4t}.$$ 

Initial conditions. Apply the prescribed initial conditions to the general solution, $x(t)$, to evaluate $c_1$ and $c_2$.

$$\begin{align*} 
x(0^-) &= 1 = c_1 + c_2 + \frac{1}{3}, \\
\dot{x}(0^-) &= -1 = -c_1 - 2c_2 - \frac{4}{3} 
\end{align*}$$

Given I.C.  

Solving the two algebraic equations simultaneously, one obtains $c_1 = \frac{5}{3}$ and $c_2 = -1$. As a result, the specific form of the general solution is

$$x(t) = \frac{5}{3} e^{-t} - e^{-2t} + \frac{1}{3} e^{-4t}$$
Example - 1.18

Example 1.18. Solve the following initial-value problem: \( \ddot{x} + \omega^2 x = 2 \sin \omega t \), \( x(0^-) = 0, \dot{x}(0^-) = 1 \). It is often encountered in the frequency response study of undamped, second-order dynamic systems.

Solution

Homogeneous solution, \( x_h(t) \). The characteristic equation and values associated with the homogeneous equation are

\[
\lambda^2 + \omega^2 = 0 \quad \Longrightarrow \quad \lambda_{1,2} = \pm j\omega
\]

which correspond to \( x_h(t) = c_1 \cos \omega t + c_2 \sin \omega t \).

Particular solution, \( x_p(t) \). The forcing function agrees with the third row of Table 1.1; and hence \( x_p(t) = A \cos \omega t + B \sin \omega t \). However, since it coincides with the homogeneous independent solutions, the particular solution is properly adjusted to \( x_p(t) = t(A \cos \omega t + B \sin \omega t) \). Substitute \( x_p(t) \) into the original ODE to obtain

\[
-2A\omega \sin \omega t + 2B \cos \omega t = 2 \sin \omega t \quad \Longrightarrow \quad A = -1, \ B = 0
\]

\[
\Longrightarrow \quad x_p(t) = -t \cos \omega t
\]
**Example -1.18 (cont)**

**General solution, \( x(t) \).**

\[
x(t) = x_h(t) + x_p(t) = c_1 \cos \omega t + c_2 \sin \omega t - t \cos \omega t.
\]

**Initial conditions.** Apply the prescribed initial conditions to the general solution, \( x(t) \), to evaluate \( c_1 \) and \( c_2 \).

\[
\begin{align*}
x(0^-) &= 0 = c_1, \\
\dot{x}(0^-) &= 1 = \omega c_2 - 1
\end{align*}
\]

Solving the two equations yields \( c_1 = 0 \) and \( c_2 = \frac{2}{\omega} \). As a result, the specific form of the general solution is

\[
x(t) = \frac{2}{\omega} \sin \omega t - t \cos \omega t
\]
**Example - 1.19**

**EXAMPLE 1.19.** Find the general solution to \( \ddot{x} + 4\dot{x} + 5x = e^{-t} \sin t \), subjected to \( x(0^-) = 0.2 \) and \( \dot{x}(0^-) = 0 \).

**Solution**

**Homogeneous solution, \( x_h(t) \).** Solving the characteristic equation yields

\[
\lambda^2 + 4\lambda + 5 = 0 \implies \lambda_{1,2} = -2 \pm j \implies x_h(t) = e^{-2t}(c_1 \cos t + c_2 \sin t)
\]

**Particular solution, \( x_p(t) \).** The forcing function is in the form suggested by the fifth row of Table 1.1, so \( x_p(t) = e^{-t}(A \cos t + B \sin t) \). Also, the two linearly independent homogeneous solutions are \( e^{-2t} \cos t \) and \( e^{-2t} \sin t \). It is then observed that neither of the two terms, \( e^{-t} \cos t \) and \( e^{-t} \sin t \), in \( x_p \) coincides with the homogeneous solutions. This implies that the suggested form of \( x_p \) is not subject to any modification. Insert \( x_p \) into the original ODE, and collect like terms, to obtain

\[
(A + 2B)e^{-t} \cos t + (B - 2A)e^{-t} \sin t
\]

\[
= e^{-t} \sin t \implies \begin{cases} A + 2B = 0 \\ B - 2A = 1 \end{cases} \implies A = -\frac{2}{5}, \quad B = -\frac{1}{5}
\]
Example 1.19 (cont)

General solution, \( x(t) \). \( x(t) = e^{-2t}(c_1 \cos t + c_2 \sin t) + \frac{1}{5}e^{-t}(-2 \cos t + \sin t) \). Application of the given initial conditions yields \( c_1 = 0.6 \) and \( c_2 = 1 \); and hence \( x(t) = e^{-2t}(0.6 \cos t + \sin t) + \frac{1}{5}e^{-t}(-2 \cos t + \sin t) \)

Referring to Eq. (1.28), these two linear combinations are rewritten as follows.

\[
0.6 \cos t + \sin t = 1.1661 \sin(t + 0.5405), \quad -2 \cos t + \sin t = \sqrt{5} \sin(t - 1.1071)
\]

Subsequently,

\[
x(t) = 1.1661e^{-2t} \sin(t + 0.5404) + 0.4472 \sin(t - 1.1071)
\]
Using Matlab to Solve Differential Equations

Matlab’s “dsolve” command is a common alternative to solving complicated differential equations by hand.

Example 1.17 will be solved again using Matlab

Example 1.17. Solve the following initial-value problem: \( \ddot{x} + 3\dot{x} + 2x = 2e^{-4t} \), \( x(0^-) = 1, \dot{x}(0^-) = -1 \).

Solution

Homogeneous solution, \( x_h(t) \). The characteristic equation and values associated with the homogeneous equation are

\[ \lambda^2 + 3\lambda + 2 = 0 \implies \lambda_{1,2} = -1, -2 \]

which correspond to \( x_h(t) = c_1e^{-t} + c_2e^{-2t} \).

Particular solution, \( x_p(t) \). The function on the right-hand side agrees with the second row of Table 1.1; and hence \( x_p(t) = ke^{-4t} \). Since this exponential function does not coincide with either of the homogeneous independent solutions, further modification is not needed. Substitute \( x_p(t) \) into the original ODE to obtain

\[ 16ke^{-4t} - 12ke^{-4t} + 2ke^{-4t} = 2e^{-4t} \implies 6ke^{-4t} = 2e^{-4t} \implies k = \frac{1}{3} \implies x_p(t) = \frac{1}{3}e^{-4t} \]

General solution, \( x(t) \). \( x(t) = x_h(t) + x_p(t) = c_1e^{-t} + c_2e^{-2t} + \frac{1}{3}e^{-4t} \).

Initial conditions. Apply the prescribed initial conditions to the general solution, \( x(t) \), to evaluate \( c_1 \) and \( c_2 \).

\[ x(0^-) = 1 = c_1 + c_2 + \frac{1}{3}, \quad \dot{x}(0^-) = -1 = -c_1 - 2c_2 - \frac{4}{3} \]

Solving the two algebraic equations simultaneously, one obtains \( c_1 = \frac{5}{3} \) and \( c_2 = -1 \). As a result, the specific form of the general solution is \( x(t) = \frac{5}{3}e^{-t} - e^{-2t} + \frac{1}{3}e^{-4t} \).
Using Matlab to Solve Differential Equations

The same solution can be obtained with Matlab and compared to the solution from Example 1.17.

\[ x(t) = \frac{5}{3}e^{-t} - e^{-2t} + \frac{1}{3}e^{-4t} \]

- clears workspace of unwanted variables
  \texttt{clear all}

- The following step will solve the differential equation from example 1.17 with the specified boundary conditions and then assign the solution (displacement) to the variable “y”.
  \texttt{y = dsolve(’5*DSx + 3*x - 27 = 2.0*exp(-4.0*t)’ , ’x(0)=1’ , ’Dx(0)=-1’ , ’t’)}

- Plots the displacement response from 0 to 6 seconds
  \texttt{plot(t, y, [0,6])}

- Assigns labels to the axes and a title to the plot
  \texttt{figure(1)}
  \texttt{title(’Displacement Response Using “dsolve”’)}
  \texttt{xlabel(’Time [s]’)}
  \texttt{ylabel(’Displacement’)}

- Comparison to given solution (see example 1.17 in dynamic systems notes)

- Define the time frame
  \texttt{t = 0:0.01:6;}

- Solution to differential equation
  \texttt{x = (5/3).*exp(-t) - exp(-2.*t) + (1/3).*exp(-4.*t)};

- Plotting of example solution
  \texttt{figure(2)}
  \texttt{plot(t,x)}
  \texttt{title(’Displacement Response From Example 1.17’)}
  \texttt{xlabel(’Time [s]’)}
  \texttt{ylabel(’Displacement’)}
Laplace Transform

Differential Equations with force or I.C. in time domain

Laplace Transform using I.C.

Algebraic Equation in Laplace domain

Rearrange terms into good form

Closed-Form Solution or Numerical Solution

Time domain \( x(t) \)

Frequency domain \( X(s) \)

Alternative: Convolution Integral (very difficult)
Laplace Transform Equation

\[ G(s) = \mathcal{L}\{g(t)\} = \int_{0}^{\infty} e^{-st} \cdot g(t) \, dt \]

**Derivatives**

\[ \mathcal{L}\{\dot{g}(t)\} = s \cdot G(s) - g(0) \]
\[ \mathcal{L}\{\ddot{g}(t)\} = s^2 \cdot G(s) - s \cdot g(0) - \dot{g}(0) \]
### Laplace Transform of several functions

<table>
<thead>
<tr>
<th>Function f(t)</th>
<th>Laplace Transform</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Unit Step</strong></td>
<td>( \frac{\mu_s}{s} )</td>
</tr>
<tr>
<td><strong>Unit Ramp</strong></td>
<td>( \frac{1}{s^2} )</td>
</tr>
<tr>
<td><strong>Unit Pulse</strong></td>
<td>( \frac{1}{s} )</td>
</tr>
<tr>
<td><strong>Unit Impulse (Dirac delta function)</strong></td>
<td>1</td>
</tr>
</tbody>
</table>

**Unit Sinusoid**

- **SIN**: \( \frac{\omega}{s^2 + \omega^2} \)
- **COS**: \( \frac{s}{s^2 + \omega^2} \)

**Exponential**

\( \frac{1}{s + a} \)
Laplace Transform

• **Unit Step**

\[ \mathcal{L}\{u_s(t)\} = u_s(s) = \int_0^\infty u_s(t) e^{-st} \, dt \]

\[ = -\frac{u_s}{s} e^{-st} \bigg|_0^\infty = -0 - \left[ -\frac{u_s}{s} \right] = \frac{u_s}{s} \]

• **Unit Ramp**

\[ \mathcal{L}\{u_s(t)\} = u_r(s) = \int_0^\infty t \, e^{-st} \, dt \]

\[ = t \frac{e^{-st}}{-s} \bigg|_0^\infty - \int_0^\infty \frac{e^{-st}}{-s} \, dt = \frac{e^{-st}}{-s^2} \bigg|_0^\infty = \frac{1}{s^2} \]

• **Inverse Laplace**

Requires manipulation of Laplace domain equation to get in a suitable form to apply \( \mathcal{L}^{-1} \).
Convolution Integral

If $G(s)$ and $H(s)$ have known inverses $g(t)$ and $h(t)$, then the product of $GH$ can be obtained by the convolution integral.

\[
\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\{G(s) \ast H(s)\} = (g \ast h)(t)
\]

\[
= \int_0^t h(\tau) g(t - \tau) \, d\tau = (h \ast g)(t)
\]

\[
= \int_0^t g(\tau) h(t - \tau) \, d\tau = (g \ast h)(t)
\]
Unit Impulse Response - First Order

• The Equation

\[ \dot{x} + \frac{1}{\tau} x = \delta(t) \quad x_0 = 0 \]

• Laplace

\[ s \cdot X(s) + \frac{1}{\tau} X(s) = 1 \]

\[ (s + \frac{1}{\tau}) X(s) = 1 \]

\[ X(s) = \frac{1}{s + \frac{1}{\tau}} \]

\[ x(t) = e^{-\frac{t}{\tau}} \]

look up inverse laplace
Laplace - First Order ODE

- The basic first order ODE can be expressed in the Laplace Domain as \( A\dot{s} + Bs = 1 \) for unit impulse and can be recast as \( A\dot{s} = 1 - Bs \)

- This can be stated as follows:
  The basic value of \( \dot{s} \) is multiplied by \( A \) -- This value is equal to 1 minus \( B \) times the integral of \( \dot{s} \)

- Normalize the equation so the coefficient on \( \dot{s} = 1 \),

\[
\therefore \quad \dot{s} = \frac{1}{A} - \frac{B}{A} \dot{s}
\]
Block Diagram - First Order ODE

Step = 1/A

Normalize to A coef

Multiply by B/A

Integrate

SCOPE

s

1

1
S

S

+ -

-
MATLAB/SIMULINK - (see tutorial)

• Simulink
• File → New Model (workspace appears)

Select the following items and place them on the worksheet

Unit Step (from Sources) - change amplitude
Sum (from Math) - need + and -
Gain (from Math) - change gain value
Integrator (from Continuous)
Scope (from Sinks)

Double click items to view or change property
GAIN block can be rotated by format.
T branch - mouse online/CTRL and right mouse button to extend line.
SIMULINK - First Order Step Response

[Diagram of a SIMULINK model for a first order step response]
SIMULINK - First Order Impulse Response
Unit Impulse Response - Second Order

• For unit impulse, \( f(t) = \delta(t), \dot{x}_0 = 0, x_0 = 0 \)

Then, \( \ddot{x} + 2\zeta \omega_n \dot{x} + \omega_n^2 x = \delta(t) \)

with \( \dot{x}_0 = 0 \) and \( x_0 = 0 \)

• Laplace with I.C. = 0

\[ (s^2 + 2\zeta \omega_n s + \omega_n^2) X(s) = 1 \]

\[ \therefore X(s) = \frac{1}{s^2 + 2\zeta \omega_n s + \omega_n^2} \]
Unit Impulse Response - Second Order

Note that (assume $\zeta < 1$)

$$s^2 + 2\zeta \omega_n s + \omega_n^2 = (s + \zeta \omega_n)^2 - (\zeta \omega_n^2) + \omega_n^2$$

$$= (s + \zeta \omega_n)^2 + \omega_n^2 (1 - \zeta^2)$$

$$= (s + \sigma)^2 + \omega_d^2$$

So that

$$X(s) = \frac{1}{(s + \sigma)^2 + \omega_d^2}$$
Unit Impulse Response - Second Order

- The inverse Laplace

\[ x(t) = \mathcal{L}^{-1} \left[ \frac{1}{\omega_d} \cdot \frac{\omega_d}{(s + \sigma)^2 + \omega_d^2} \right] \]

\[ = \frac{1}{\omega_d} e^{-\sigma t} \sin \omega_d t \]

\[ = \frac{1}{\omega_d} e^{-\zeta \omega t} \sin \omega_d t \]

If I.C. \( \neq 0 \), then a more involved (but possible solution) exists.
**Unit Step Response - Second Order**

- **For unit step** \( f(t) = u(t), \quad x_0 = 0, \quad \dot{x}_0 = 0 \)

\[ \ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = u(t) \]

with \( \dot{x}_0 = 0, \quad x_0 = 0 \)

- **Laplace with IC=0 gives**

\[ (s^2 + 2\zeta\omega_n s + \omega_n^2) x(s) = \frac{1}{s} \]

\[ \therefore \quad X(s) = \frac{1}{s} \cdot \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2} \]
Unit Step Response - Second Order

• Again assume $\zeta < 1$ - then

$$X(s) = \frac{1}{\omega_n^2} \left[ \frac{1}{s} - \frac{s + 2\zeta \omega_n}{s^2 + 2\zeta \omega_n s + \omega_n^2} \right]$$

• Then the inverse Laplace

$$X(t) = \mathcal{L}^{-1} \left[ \frac{1}{\omega_n^2} \left[ \frac{1}{s} - \frac{s + 2\zeta \omega_n}{s^2 + 2\zeta \omega_n s + \omega_n^2} \right] \right]$$

$$X(t) = \frac{1}{\omega_n^2} \left[ 1 - e^{-\zeta \omega_n t} \left[ \cos \omega_d t + \frac{\zeta}{\sqrt{1 - \zeta^2}} \sin \omega_d t \right] \right]$$
**Example 1.40.** Consider the mechanical system shown in Fig. 1.36, consisting of a block of mass $m$, a linear spring of stiffness $k$, and where the coefficient of viscous damping is $b$. The system is subjected to an applied force $f(t) = u_s(t)$ and zero initial conditions. Determine the displacement, $x(t)$, at any time $t > 0$. All parameters are in consistent physical units!
Example - 1.40 (cont)

**Solution.** The equation of motion for this system, derived in Chapter 4, is \( m\ddot{x} + b\dot{x} + kx = f(t) \). Substituting for the numerical values of the physical parameters, the initial-value problem is

\[
\ddot{x} + 2\dot{x} + 2x = u_s(t), \quad x(0^-) = 0 = \dot{x}(0^-)
\]

Comparing with the standard form of second-order systems, defined by Eq. (1.22), we have

\[
\begin{align*}
2\zeta \omega_n &= 2 \\
\omega_n^2 &= 2
\end{align*}
\implies \begin{cases} 
\omega_n = \sqrt{2} \\
\zeta = \frac{\sqrt{2}}{2} < 1
\end{cases}
\]

Following the results of case (1) above, the unit-step response is given by Eq. (1.74), as

\[
x(t) = \frac{1}{\omega_n^2} \left[ 1 - \frac{1}{\sqrt{1 - \zeta^2}} e^{-\zeta \omega_n t} \sin(\omega_d t + \phi) \right], \quad \text{where } \phi = \tan^{-1}\left(\frac{\sqrt{1 - \zeta^2}}{\zeta}\right)
\]

Substitution of the damping ratio, as well as the damped and undamped natural frequencies, yields

\[
x(t) = \frac{1}{2} \left[ 1 - \sqrt{2} e^{-t} \sin\left(t + \frac{\pi}{4}\right) \right]
\]
General Laplace Formulation

\[ \ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = f(t) \]

- \textbf{Laplace}

\[
\mathcal{L}(\ddot{x}) = s^2 x(s) - sx(0) - \dot{x}(0) \\
\mathcal{L}(\dot{x}) = sx(s) - x(0) \\
\mathcal{L}(x) = X(s) \\
\mathcal{L} F(t) = F(s)
\]

\[
s^2 x(s) - sx(0) - \dot{x}(0) + 2\zeta\omega_n \left[ sx(s) - x(0) \right] + \omega_n^2 x(s) = F(s)
\]
**General Laplace Formulation**

\[
(s^2 + 2 \zeta \omega_n s + \omega_n^2)x(s) = F(s) + (s + 2 \zeta \omega_n)x(0) + \dot{x}(0)
\]

**Applied Force and Initial Conditions**

\[
x(s) = \frac{F(s)}{s^2 + 2 \zeta \omega_n s + \omega_n^2} + \frac{(s + 2 \zeta \omega_n)x(0) + \dot{x}(0)}{s^2 + 2 \zeta \omega_n s + \omega_n^2}
\]

*If initial conditions are zero, then the system transfer function is*

\[
\frac{x(s)}{F(s)} = H(s) = \frac{1}{s^2 + 2 \zeta \omega_n s + \omega_n^2}
\]

*Many books use \( G(s) \)!*

aka \( \frac{1}{ms^2 + cs + k} \)
SIMULINK - 2nd Order Impulse Response (or RAMP)

\[ 100\ddot{x} + 40\dot{x} + 1000x = F(t) \]

OR...
**SIMULINK – 2\(^{nd}\) Order Impulse Response (or RAMP)**

**In alternate form (using the Transfer Function Block)**

\[ 100\ddot{x} + 40\dot{x} + 1000x = F(t) \]
SIMULINK – 2nd Order Impulse Response (or RAMP)

General Equation is

\[ \ddot{x} = \frac{1}{m} \left[ F(t) - kx - cx \right] \]

\[ m\ddot{x} + c\dot{x} + kx = F(t) \]

This example is

\[ 100\ddot{x} + 40\dot{x} + 1000x = F(t) \]