Mechanical Systems

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Mechanical Systems - Translational Mass Element

Translation of a particle moving in space due to an applied force is given by:

\[ f = \frac{dp}{dt} \]

Where:  
\[ f = \text{force} \]
\[ p = \text{momentum} = mv \]

Considering the mass to be constant:

\[ f = \frac{d(mv)}{dt} \Rightarrow f dt = m dv \Rightarrow f = m \frac{dv}{dt} = ma \]
Mechanical Systems - Translational Mass Element

Displacement, velocity, and acceleration are all related by time derivatives as:

\[ a = \frac{dv}{dt} = \frac{d^2x}{dt^2} \]

\[ a = \dot{v} = \ddot{x} \]
Mechanical Systems - Rotational Mass Element

Centroidal mass moment of inertia - \( I_c \) (not to be confused with \( I \) - area moment of inertia used in strength of materials)

Angular acceleration

\[
\alpha = \frac{d\omega}{dt} = \dot{\omega}
\]

where:

\( \omega = \text{angular velocity} \)

\( \theta = \text{angular displacement} \)

Then:

\[
\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt}
\]
**Mechanical Systems - Translational Spring Element**

A linear spring is considered to have no mass described by:

\[
f_k = kx_{rel} = k(x_1 - x_2)
\]

*(Torsional spring follows the same relationship)*
**Translational Spring Element**

- **Hardening Spring**
- **Linear**
- **Softening Spring**

**Bi-Linear**

\[ f \]

\[ x \]

**Gap**

\[ f \]

\[ x \]

**Cubic**

\[ f \]

\[ x \]

\[ k = \text{lb/in} \]

\[ = \text{N/m} \]
**Damper Element**

*Viscous (fluid), Coulomb (dry friction), and structural damping (hysteretic)*

**Viscous Dashpot**

\[ f_c = c v_{rel} \]

**Coulomb Damper**

In order to have motion, the applied force must overcome the static friction. As soon as sliding occurs, the dynamic friction becomes appropriate.

\[ f_c = c(v_1 - v_2) \]
Equivalence - Springs in Parallel

Both springs see the same displacement

\[ f = f_1 + f_2 \]

\[ k_{eq}x = k_1x + k_2x \]

\[ \therefore k_{eq} = k_1 + k_2 \]
Equivalence - Springs in Series

Both springs see the same force but different displacements

\[ \delta = \delta_1 + \delta_2 \]
\[ \frac{f}{k_{eq}} = \frac{f_1}{k_1} + \frac{f_2}{k_2} \]

But \[ f = f_1 = f_2 \]

\[ f = k_1(x_1-x_2) = k_1 \delta_1 \]
\[ f_2 = k_2 x_2 = k_2 \delta_2 \]

\[ \frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} \]

\[ k_{eq} = \frac{k_1 k_2}{k_1 + k_2} \]
Translational Systems

Newton's Second Law - THE RIGHT WAY

\[ \sum F = ma \quad \text{OR} \quad \sum F_x = m a_x \quad \rightarrow^+ \]
\[ \sum F_y = m a_y \quad \uparrow^+ \]

Note that this applies to the center of mass which is not necessarily the center of gravity.

Free-Body Diagram & Sign Convention
**Translational Systems - Newton's 2nd Law**

*Assume spring and dashpot are stretched*

\[
\sum F_x = ma_x
\]

\[
f(t) - F_c - F_k = m\ddot{x}
\]

**OR**

\[
f(t) - c\dot{x} - kx = m\ddot{x}
\]

*or in standard input-output differential form*

\[
m\dddot{x} + c\ddot{x} + kx = f(t) \iff \dddot{x} + \frac{c}{m}\ddot{x} + \frac{k}{m}x = \frac{f(t)}{m}
\]

\[
\dddot{x} + 2\zeta\omega_n\ddot{x} + \omega_n^2 x = f(t)
\]

\[
\begin{align*}
\zeta &= \frac{c}{c_c} \\
c_c &= 2m\omega_n \\
\omega_n^2 &= \frac{k}{m}
\end{align*}
\]

- **damping ratio**
- **critical damping**
- **natural frequency**
D’Alembert’s Principle – The Fictitious Force

The mass times acceleration is sometimes described as a ‘fictitious force’, ‘reverse effective force’ or ‘apparent force’

\[ \sum F + (-ma) = 0 \]

Initially developed since it looks like a classical force balance – but often confuses many students.

**DO NOT USE D’ALEMBERT!!!!!**
**USE NEWTON’S SECOND LAW**
Example - Pendulum Problem

Mass at end of massless string

\[ J = ml^2 \]

\[ \begin{align*}
J\ddot{\theta} &= -mg\sin\theta \\
\text{OR} \quad ml^2\ddot{\theta} + mg\sin\theta &= 0
\end{align*} \]

Then

\[ \ddot{\theta} + \frac{g}{l}\sin\theta = 0 \]

\[ \begin{align*}
\text{for small } \theta, \quad &\ddot{\theta} + \frac{g}{l}\theta = 0 \\
\text{Nat'l freq.} \quad &\omega_n = \sqrt{\frac{g}{l}}
\end{align*} \]
Example – Differential Equation about Equilibrium

\[ \sum F_y = ma_y \]
\[ -ky + mg = m\ddot{y} \]
\[ -k(x_{st} + x) + mg = m\ddot{x} \]
\[ \text{but } mg = kx_{st} \]
\[ \therefore m\ddot{x} + kx = 0 \]

Source: Dynamic Systems - Vu & Esfandiari

Therefore, the equations can be written about the equilibrium point and the effect of gravity makes no difference.
Systems with Displacement Input

\[ \sum F_x = m a_x \]

\[ FBD \]

\[ k(y-x) + c(\dot{y} - \dot{x}) = m\ddot{x} \quad \text{OR} \quad m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky \]

In terms of natural frequency and damping ratio

\[ \ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = 2\zeta\omega_n \dot{y} + \omega_n^2 y \]

The force exerted can be found to be

\[ f(t) = k(y-x) + c(\dot{y} - \dot{x}) \]
Transfer Function and State Space

\[ m\ddot{x} + c\dot{x} + kx = f(t) \]

\[ \mathcal{L}(m\ddot{x}) = s^2mX(s) - msX_0 - m\dot{X}_0 \]
\[ \mathcal{L}(kx) = kX(s) \]
\[ \mathcal{L}(c\dot{x}) = scX(s) - cX_0 \]
\[ \mathcal{L}(f(t)) = f(s) \]

\[ s^2mX(s) - msX_0 - m\dot{X}_0 + scX(s) - cX_0 + kX(s) = f(s) \]
Transfer Function and State Space

Grouping and rearranging:

\[(ms^2 + cs + k)X(s) = f(s) + (ms + c)X_0 + m\dot{X}_0\]

Assume initial conditions are zero and rearranging terms to obtain OUT/IN form

Then:

\[H(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2 + cs + k}\]

Sometimes written with

\[b(s) = ms^2 + cs + k\]
Frequency Response Function - SDOF

The frequency response function is the system transfer function evaluated along $s = j\omega$

Recall:

$$h(s) = \frac{1}{ms^2 + cs + k}$$

The complex valued function defines the surface shown

Source: Vibrant Technology
**SDOF – Transfer Function**

**Polynomial Form**

\[ h(s) = \frac{1}{ms^2 + cs + k} \]

**Pole-Zero Form**

\[ h(s) = \frac{1/m}{(s-p_1)(s-p_1^*)} \]

**Partial Fraction Form**

\[ h(s) = \frac{a_1}{s-p_1} + \frac{a_1^*}{s-p_1^*} \]

**Exponential Form**

\[ h(t) = \frac{1}{m\omega_d} e^{-\zeta\omega t} \sin \omega_d t \]
SDOF – Frequency Response Function

**Polynomial Form**

\[ h(j\omega) = \frac{1}{-m\omega^2 + cj\omega + k} \]

**Pole-Zero Form**

\[ h(j\omega) = \frac{1/m}{(j\omega - p_1)(j\omega - p_1^*)} \]

**Partial Fraction Form**

\[ h(j\omega) = \frac{a_1}{(j\omega - p_1)} + \frac{a_1^*}{(j\omega - p_1^*)} \]
**SDOF - Transfer Function**

Transfer Function approach is used extensively in design but is limited to linear, time-invariant systems.

1. **T.F.** - method to express output relative to input
2. **T.F.** - system property - independent of the nature of excitation
3. **T.F.** contains necessary units but does not provide physical structure of system
4. If **T.F.** is known, then response can be evaluated due to various inputs
5. If **T.F.** is unknown, it can be established experimentally by measuring output response due to known measured inputs
S-plane Plots

\[ \sigma + j \omega \]

\[ \zeta = 0 \]
\[ \zeta = 0.1 \]
\[ \zeta = 0.3 \]
\[ \zeta = 0.7 \]
\[ \zeta = 1.0 \]

UNSTABLE
STABLE
Experimental Determination of Damping Ratio

Determine decay of amplitude $x_1$ at $t_1$ and again at $n$ cycles later $x_n$ at $t_1 + (n-1)T$

Then

$$\frac{x_1}{x_2} = \frac{e^{-\zeta \omega_n t}}{e^{-\zeta \omega_n (t+T)}} = \frac{1}{e^{-\zeta \omega_n T}} = e^{\zeta \omega_n T}$$

OR

$$\frac{x_1}{x_n} = \frac{1}{e^{-\zeta \omega_n (n-1)T}} = e^{(n-1)\zeta \omega_n T}$$
Log Decrement

\[
\ln\left(\frac{x_1}{x_2}\right) = \frac{1}{n-1} \ln\left(\frac{x_1}{x_n}\right) = \zeta \omega_n T
\]

\[
= \zeta \omega_n \frac{2\pi}{\omega_d} = \frac{2\pi \zeta}{\sqrt{1 - \zeta^2}}
\]

\[
\therefore \frac{1}{n-1} \ln\left(\frac{x_1}{x_n}\right) = \frac{2\pi \zeta}{\sqrt{1 - \zeta^2}}
\]

For damping < 10%

\[
\ln \frac{x_1}{x_2} \approx 2\pi \zeta
\]

Note: This damping ratio formulation is applicable to any 2\textsuperscript{nd} order system of this form
Estimate of Response Time

The response of a mechanical system due to an initial displacement is given as:

\[ x(t) = \frac{X_0}{\sqrt{1-\zeta^2}} \, e^{-\zeta \omega_n t} \cos(\omega_d t - \phi) \]

The exponential response envelope is

\[ \frac{X_0}{\sqrt{1-\zeta^2}} \, e^{-\zeta \omega_n t} \]

whose time constant \( T \) of the exponential is

\[ \frac{1}{\zeta \omega_n} = \frac{1}{\sigma} \]
**Estimate of Response Time**

The response of the second-order system in terms of the settling time is

\[
t_s = 4T = \frac{4}{\zeta \omega_n} = \frac{4}{\sigma}
\]

which will cause 2% of the initial value

Source: Dynamic Systems - Vu & Esfandiari
State Space Representation

The 'state' of the system can be described in terms of the displacement and velocity as

\[
\begin{bmatrix}
\frac{x_1}{x_2}
\end{bmatrix} = \begin{bmatrix}
\frac{x}{\dot{x}}
\end{bmatrix} \quad \rightarrow \quad X = \begin{bmatrix}
\frac{x_1}{x_2}
\end{bmatrix} = \begin{bmatrix}
\frac{x(\text{displ})}{\dot{x}(\text{velocity})}
\end{bmatrix}
\]

\(u = f(\text{force})\) and \(y = x(\text{measured by sensor})\)

Then

\[
\ddot{x} = -\frac{k}{m}x - \frac{c}{m}\dot{x} + \frac{1}{m}f(t)
\]

OR

\[
\dot{x}_2 = -\frac{k}{m}x_1 - \frac{c}{m}x_2 + \frac{1}{m}u
\]
State Space Representation

So that the state space representation is

\[
\begin{align*}
\text{State Equation} & \quad \left\{ \begin{array}{c} \dot{x}_1 \\ \dot{x}_2 \end{array} \right\} = \left[ \begin{array}{cc} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{array} \right] \left\{ \begin{array}{c} x_1 \\ x_2 \end{array} \right\} + \left\{ \begin{array}{c} 0 \\ \frac{1}{m} \end{array} \right\} u \\
\text{Output Equation} & \quad y = \left[ \begin{array}{cc} 1 & 0 \end{array} \right] \left\{ \begin{array}{c} x_1 \\ x_2 \end{array} \right\} + 0 \cdot u
\end{align*}
\]
Lagrange’s Equations

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial q_i} \right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_{nci}
\]

\(T\) - Kinetic energy
\(V\) - Potential energy
\(Q_{nci}\) - non-conservative generalized forces
\(q_i\) - independent generalized coordinates
\(n\) - total # independent generalized coordinates

Kinetic Energy is a function of \(T(q_i,t)\) \(dq/dt\)
Potential energy is the sum of elastic potential \(V_e\)
and gravitational potential \(V_g\)
Potential Energy is a function of \(V(q_i,t)\)
Lagrange’s Equations

One standard form of Lagrange’s Equation

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = Q_{nci} \quad \text{where } L = T - V
\]

We can then write

\[
\frac{d}{dt} \left( \frac{\partial (T - V)}{\partial \dot{q}_i} \right) - \frac{\partial (T - V)}{\partial q_i} = Q_{nci}
\]

(Note V is not a function of \( \dot{q}_i \))
Lagrange’s Equations

Kinetic energy for a particle

1D ⇒ \( T = \frac{1}{2}mv^2 \)

2D ⇒ \( T = \frac{1}{2}mv^2 + \frac{1}{2}I_c \omega^2 \)

If the mass is not located at a point (such as a particle), then a more complicated form of these equations is necessary

Potential Energy of an elastic element is

\( V_e = \frac{1}{2}k\delta^2 \)

Potential Energy of a mass is

\( V_g = mgh \)
Lagrange's Equations

Non-conservative Forces are those that cannot be derived from a potential function (i.e., external forces, frictional forces)

Generalized Forces are given by Virtual Work.

\[ \delta W = \sum Q_i \delta q_i = Q_1 \delta q_1 + Q_2 \delta q_2 + \cdots \]

To determine \( Q_j \), obtain \( \delta W \), then let all \( \delta q_i = 0 \) except \( \delta q_j \)

Thus

\[ Q_j = \frac{\delta W}{\delta q_j} \quad \left\{ \begin{array}{c} \delta q_j = 0 \\ j \neq i \end{array} \right. \]
Lagrange’s Equations

Non-conservative Forces are then:

\[ \delta W_{nc} = \sum Q_{nci} \delta q_i = Q_{nc1} \delta q_1 + Q_{nc2} \delta q_2 + \cdots \]

Then

\[ Q_{nci} = \frac{\delta W_{nc}}{\delta q_i} \begin{cases} \delta q_i = 0 \\ j \neq i \end{cases} \]
Example using Lagrange Equation

Use Lagrange EQ to obtain differential equation for SDOF system

Only one independent generalized coordinate exists: \( q=x \)

\[
\begin{align*}
\text{Kinetic Energy} & \quad T = \frac{1}{2} m v_c^2 = \frac{1}{2} m \dot{x}^2 \\
\text{Potential Energy} & \quad V = V_e = \frac{1}{2} k x^2 \quad (V_g = 0) \\
\text{Non-conservative Forces} & \quad f(t) - \text{applied} \\
& \quad c \dot{x} - \text{dissipative}
\end{align*}
\]
Example using Lagrange Equation

Non-Conservative Forces

\[ \delta W_{nc} = [f(t) - c\dot{x}] \delta x \]

\[ \delta W_{nc} = Q_{nc} \delta x \]

\[ Q_{nc} = \frac{\delta W_{nc}}{\delta x} = f(t) - c\dot{x} \]

Lagrange Equation

\[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{x}} \right) - \frac{\partial T}{\partial x} + \frac{\partial V}{\partial x} = Q_{nc} \]
Example using Lagrange Equation

Lagrange Equation

\[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{x}} \right) - \frac{\partial T}{\partial x} + \frac{\partial V}{\partial x} = Q_{nc} \]

Where

\[ \frac{\partial T}{\partial \dot{x}} = \frac{\partial}{\partial \dot{x}} \left( \frac{1}{2} m \dot{x}^2 \right) = m \ddot{x} \]

\[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{x}} \right) = m \ddot{x} \]

\[ \frac{\partial T}{\partial x} = \frac{\partial}{\partial x} \left( \frac{1}{2} m \dot{x}^2 \right) = 0 \]

\[ \frac{\partial V}{\partial x} = \frac{\partial}{\partial x} \left( \frac{1}{2} kx^2 \right) = kx \]

Then substituting

\[ m \ddot{x} + kx = f(t) - c \dot{x} \]

OR

\[ m \ddot{x} + c \dot{x} + kx = f(t) \]
Example using Lagrange Equation

Now let's repeat this with the Lagrange function

\[
L = T - V = \frac{1}{2} m \ddot{x}^2 - \frac{1}{2} kx^2
\]

\[
\frac{\partial}{\partial t} \left( \frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = Q_{nc}
\]

\[
\frac{\partial L}{\partial \dot{x}} = \frac{\partial}{\partial \dot{x}} \left( \frac{1}{2} m \dot{x}^2 - kx^2 \right) = m \ddot{x}
\]

\[
\frac{\partial L}{\partial \dot{x}} = m \ddot{x}
\]

\[
\frac{\partial}{\partial t} \left( \frac{\partial L}{\partial \dot{x}} \right) = m \dddot{x}
\]

\[
\therefore m \dddot{x} + c \ddot{x} + kx = f(t)
\]
Example - Translational Mechanical System

Solution

(a) The datum (reference) for $V_g$ for $y$ is shown in Fig. 4.57. This system has only one independent generalized coordinate, namely $q = y$. The kinetic energy is

$$T = \frac{1}{2} m v_c^2 = \frac{1}{2} m \dot{y}^2$$

The potential energy is

$$V = V_c + V_g = \frac{1}{2} k \delta^2 + mgh = \frac{1}{2} k y^2 + mg(-y)$$

The nonconservative virtual work $\delta W_{nc}$ is zero:

$$\delta W_{nc} = 0$$

Source: Dynamic Systems - Vu & Esfandiari
Example – Translational Mechanical System

The Lagrange’s equation becomes

\[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{y}} \right) - \frac{\partial T}{\partial y} + \frac{\partial V}{\partial y} = 0 \]

We have

\[ \frac{\partial T}{\partial \dot{y}} = \frac{\partial}{\partial \dot{y}} \left( \frac{1}{2} m \dot{y}^2 \right) = m \ddot{y} \quad \Rightarrow \quad \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{y}} \right) = m \ddot{y} \]

\[ \frac{\partial T}{\partial y} = \frac{\partial}{\partial y} \left( \frac{1}{2} m \dot{y}^2 \right) = 0 \]

\[ \frac{\partial V}{\partial y} = \frac{\partial}{\partial y} \left( \frac{1}{2} ky^2 - mg y \right) = ky - mg \]

Thus,

\[ m \dddot{y} + ky - mg = 0 \]

The differential equation is

\[ m \dddot{y} + ky = mg \]

(b) The datum (reference) for \( V_g \) for \( x \) is shown in Fig. 4.58. The kinetic energy is

\[ T = \frac{1}{2} m v_c^2 = \frac{1}{2} m x^2 \]

Source: Dynamic Systems - Vu & Estandiari
Example - Translational Mechanical System

The potential energy is

\[ V = V_c + V_g = \frac{1}{2}k\delta^2 + mgh = \frac{1}{2}k(x_{st} + x)^2 + mg(-x) \]

Source: Dynamic Systems - Vu & Esfandiari
Example - Translational Mechanical System

The nonconservative virtual work $\delta W_{nc}$ is zero:

$$\delta W_{nc} = 0$$

The Lagrange's equation becomes

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{x}} \right) - \frac{\partial T}{\partial x} + \frac{\partial V}{\partial x} = 0$$

We have

$$\frac{\partial T}{\partial \dot{x}} = \frac{\partial}{\partial \dot{x}} \left( \frac{1}{2} m \dot{x}^2 \right) = m \ddot{x} \implies \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{x}} \right) = m \dddot{x}$$

$$\frac{\partial T}{\partial x} = \frac{\partial}{\partial x} \left( \frac{1}{2} m \dot{x}^2 \right) = 0$$

$$\frac{\partial V}{\partial x} = \frac{\partial}{\partial x} \left[ \frac{1}{2} k(x_{st} + x)^2 - mgx \right]$$

$$= k(x_{st} + x)(1) - mg = kx_{st} - mg + kx = kx$$

because at static equilibrium $kx_{st} = mg$. Finally, the differential equation is

$$m \dddot{x} + kx = 0$$

where the term $mg$ does not enter into the equation. We can conclude that it is simpler to represent the mathematical model in terms of $x$, which is the displacement measured from the static equilibrium position. Thus, gravity may be ignored for this type of mass-spring system.

Source: Dynamic Systems - Vu & Esfandiari
Example - Two DOF Systems

Consider

$$f_1(t) \rightarrow m_1 \rightarrow x_1 \leftarrow k(x_1 - x_2) \rightarrow m_2 \rightarrow f_2$$

FBD (assume $x_1 > x_2$)

$$\sum F = ma(1) f_1(t) - c(\ddot{x}_1 - \ddot{x}_2) - k(x_1 - x_2) = m_1 \ddot{x}_1$$

$$\sum F = ma(2) f_2(t) + c(\ddot{x}_1 - \ddot{x}_2) + k(x_1 - x_2) = m_2 \ddot{x}_2$$

STATE ASSUMPTIONS!!!
Example - Two DOF Systems

Rearranging Terms

\[ m_1 \ddot{x}_1 + c \dot{x}_1 - c \dot{x}_2 + kx_1 - kx_2 = f_1(t) \]

\[ m_2 \ddot{x}_2 - c \dot{x}_1 + c \dot{x}_2 - kx_1 + kx_2 = f_2(t) \]

\[
\begin{bmatrix}
  m_1 & 0 \\
  0 & m_2
\end{bmatrix}
\begin{bmatrix}
  \ddot{x}_1 \\
  \ddot{x}_2
\end{bmatrix}
+ \begin{bmatrix}
  c & -c \\
  -c & c
\end{bmatrix}
\begin{bmatrix}
  \dot{x}_1 \\
  \dot{x}_2
\end{bmatrix}
+ \begin{bmatrix}
  k & -k \\
  -k & k
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2
\end{bmatrix} = \begin{bmatrix}
  f_1 \\
  f_2
\end{bmatrix}
\]
Example - Two DOF Systems

Consider

\[ f_1(t) \rightarrow m_1 \rightarrow c(\ddot{x}_2 - \ddot{x}_1) \leftarrow m_2 \rightarrow f_2(t) \]

FBD (assume \( x_2 > x_1 \))

\[ \sum F = ma(1) \quad f_1(t) - k_1x_1 + c(\ddot{x}_2 - \ddot{x}_1) + k_2(x_2 - x_1) = m_1\dddot{x}_1 \]

\[ \sum F = ma(2) \quad f_2(t) - c(\ddot{x}_2 - \ddot{x}_1) - k_2(x_2 - x_1) = m_2\dddot{x}_2 \]
Example - Two DOF Systems

Rearranging terms

\[ m_1 \ddot{x}_1 + c \dot{x}_1 - c \dot{x}_2 + (k_1 + k_2)x_1 - k_2x_2 = f_1(t) \]
\[ m_2 \ddot{x}_2 - c \dot{x}_1 + c \dot{x}_2 - k_2x_1 + k_2x_2 = f_2(t) \]

\[
\begin{bmatrix}
  m_1 & 0 \\
  0 & m_2 \\
\end{bmatrix}
\begin{bmatrix}
  \ddot{x}_1 \\
  \ddot{x}_2 \\
\end{bmatrix}
+
\begin{bmatrix}
  c & -c \\
  -c & c \\
\end{bmatrix}
\begin{bmatrix}
  \dot{x}_1 \\
  \dot{x}_2 \\
\end{bmatrix}
+
\begin{bmatrix}
  (k_1 + k_2) & -k_2 \\
  -k_2 & k_2 \\
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2 \\
\end{bmatrix}
=
\begin{bmatrix}
  f_1 \\
  f_2 \\
\end{bmatrix}
\]
Rotational Systems

A rotational system follows the same equations developed for translation

Newton’s Second Law

\[ \sum M_0 = I_0 \alpha \]

- \( M_0 \) – moments applied
- \( I_0 \) – mass moment of int
- \( \alpha \) – \( \dot{\omega} \) angular accel.

Mass moment of inertia of rigid body about axis

\[ J \text{ or } I_0 = \int r^2 \, dm \]
Rotational Systems

Torsion spring stiffness similar to translation

\[ T_k = K_T (\theta_2 - \theta_1) \]

\[ T_k = K_T (\theta_{\text{REL}}) \]

Dashpot similar to translation

\[ T_D = C_D (\dot{\theta}_2 - \dot{\theta}_1) \]

\[ T_D = C_D (\theta_{\text{REL}}) \]

Right hand rule convention determines +/-
Rotational Systems

\[ \theta_{IN} \rightarrow J \rightarrow \theta_{OUT} \]

**FBD**

\[ T_S \quad J \quad T_D \]
Rotational Systems - Equations

\[ \sum M = J\alpha \Rightarrow T_S + T_D = J \frac{d^2\theta}{dt^2} = I\alpha \]

\[ T_S = K_T (\theta_{IN} - \theta_{OUT}) \]

\[ T_D = -B\dot{\theta}_{OUT} \]

\[ K_T (\theta_{IN} - \theta_{OUT}) - B\dot{\theta}_{OUT} = J\ddot{\theta}_{OUT} \]

\[ J\ddot{\theta}_{OUT} + B\dot{\theta}_{OUT} + K_T \theta_{OUT} = K_T \theta_{IN} \]

Most systems we will treat will be 2D or planar systems. Modeling of general 3D bodies is more complex and beyond the scope of this course.
Example - SDOF Torsional System

A torsional system: (a) physical system, (b) FBD
Example - SDOF Torsional System

Consider a single-degree-of-freedom (SDOF) torsional system. The system consists of a shaft of torsional stiffness $K$, a disk of mass-moment of inertia $J$, and a torsional damper $B$. Derive the differential equation.

Solution. Applying the moment equation about the mass center along the longitudinal axis.

$$ + \sum M_c = I_c \alpha $$

This sign convention is simpler and useful for the given angle $\theta$. Thus,

$$ T(t) - K\theta - B\dot{\theta} = J\ddot{\theta} $$

The differential equation in the input-output form is

$$ J\ddot{\theta} + B\dot{\theta} + K\theta = T(t) $$
Example - Two DOF Torsional System

Example 4.15. Consider the TDOF torsional system shown in Fig. 4.40. The system consists of a shaft of torsional stiffness $K$, two disks of polar moments of inertia $J_1$ and $J_2$, and a torsional damper $B$. Draw the necessary free-body diagrams and derive the differential equations. Then express the equations in the second-order matrix form.

Source: Dynamic Systems - Vu & Esfandiari
Example - Two DOF Torsional System

Solution. Assuming $\theta_1 > \theta_2 > 0$, the free-body diagrams of the disks are shown in Fig. 4.41. Applying the moment equation about the mass centers along the longitudinal axis,

$$ + \sum M_c = I_c \alpha + $$

Disk 1:

$$ T_1(t) - K(\theta_1 - \theta_2) - B(\dot{\theta}_1 - \dot{\theta}_2) = J_1\ddot{\theta}_1 $$

Disk 2:

$$ T_2(t) - K(\theta_1 - \theta_2) + B(\dot{\theta}_1 - \dot{\theta}_2) = J_2\ddot{\theta}_2 $$

The differential equations of the system are expressed in the standard input-output form as

$$ J_1\ddot{\theta}_1 + B\dot{\theta}_1 - B\dot{\theta}_2 + K\theta_1 - K\theta_2 = T_1(t) $$

$$ J_2\ddot{\theta}_2 - B\dot{\theta}_1 + B\dot{\theta}_2 - K\theta_1 + K\theta_2 = T_2(t) $$

and in the second-order matrix form as

$$ \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} B & -B \\ -B & B \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} K & -K \\ -K & K \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} T_1(t) \\ T_2(t) \end{bmatrix} $$

Note that this torsional system is analogous to the corresponding TDOF translational system (Fig. 4.28).

Source: Dynamic Systems - Vu & Esfandiari
Example - Rigid Body in Planar Motion

Point mass on string - Moment method

\[ \sum M_0 = I_0 \alpha \]

\[-L \sin \theta \ mg = (0 + ml^2) \ddot{\theta} \]

\[ mL^2 \ddot{\theta} + mgL \sin \theta = 0 \]

As before \[ \ddot{\theta} + \frac{g}{L} \sin \theta = 0 \]
Example – Pendulum Problem

Thin uniform rod of mass m and length l is a pendulum

\[ \sum M_0 = I_0 \alpha \]

\[ -\frac{L}{2} \sin \theta \ mg = I_0 \ddot{\theta} \]

Then

\[ I_0 \ddot{\theta} + mg \frac{1}{2} \sin \theta = 0 \]
Example – Pendulum Problem

**Linearization:** For small $\theta \rightarrow \sin \theta \Rightarrow \theta$

\[
I_0 \ddot{\theta} + \frac{mgL}{2} \theta = 0
\]

or
\[
\ddot{\theta} + \frac{mgL}{2I_0} \theta = 0
\]

*where*
\[
I_0 = I_c + md^2
\]
\[
= I_c + m\left(\frac{L}{2}\right)^2
\]
\[
I_0 = \frac{1}{3} mL^2
\]
\[
I_c = \frac{1}{12} mL^2
\]
Mixed Translation and Rotation

Pulley system

$I_c$ of pulley, mass $m$ of radius $r$,
Tension in string

Newton's second law for mass $m$ \[ m\ddot{x} = -T \]
(everything measured from equilibrium so no $mg$ term)

Rotation of pulley \[ J\ddot{\theta} = Tr - krx \]

For small angle $x = r\theta$ then \[ J\ddot{\theta} = -m\ddot{x}r - krx \]

and \[ (J + mr^2)\ddot{\theta} + kr^2\theta = 0 \]

or \[ \ddot{\theta} + \frac{kr^2}{J + mr^2} \theta = 0 \]

The natural frequency is \[ \omega_n = \sqrt{\frac{kr^2}{J + mr^2}} \]
Example - Cart-Pendulum Problem

Consider the pendulum system shown attached to a horizontal cart.

Cart moves horizontally on frictionless surface. Mass on inextensible string.

This is a mixed problem. First solve the pendulum and then the cart translation.

The general moment about point P (where the string is attached to the cart mass) is needed to sum the forces for Newton’s Second Law.
Example - Cart-Pendulum Problem

\[ \sum M_p = I_p \alpha + m r_{c/p} \times a_p \]

\[ P_y \]
\[ P_x \]
\[ L \sin \theta \]
\[ L \sin \theta \]
\[ mg \]
\[ a_p = \ddot{x} \]
\[ r_{c/p} = L \]
\[ C_i \]
Example – Cart-Pendulum Problem

The cross product term is
\[ r_{c/p} \times a_p = \left| r_{c/p} \right| a_p \sin \phi \]
\[ = L\ddot{x}\sin(90 - \theta) = L\ddot{x}\cos \theta \]

Using the parallel axis theorem, the mass of the pendulum at a distance \( L \) gives
\[ I_p = mL^2 \Rightarrow I_p \alpha = mL^2\ddot{\theta} \]

The moment about \( P \) due to the mass on the pendulum is
\[ M = -mgL\sin \theta \]

The general moment equation becomes
\[ -mgL\sin \theta = mL^2\ddot{\theta} + mL\ddot{x}\cos \theta \]
Example - Cart-Pendulum Problem

Now the translational equation is evaluated. Only horizontal is considered.

\[ M \rightarrow f(t) \]

\[ \theta \]

\[ L \sin \theta \]

\[ kx \]

\[ cx \]

\[ M \]

\[ a_{px} = \ddot{x} \]

\[ \text{centripetal} \]

\[ \text{tangential} \]

\[ L\dot{\theta}^2 \sin \theta \]

\[ L\dot{\theta} \cos \theta \]

\[ L\dot{\theta} \]

\[ \theta \]

\[ L \]

\[ m \]
Example – Cart-Pendulum Problem

For the cart, Newton's Second Law

The acceleration of the pendulum mass

\[
\ddot{x} + L\ddot{\theta}\cos\theta - L\dot{\theta}^2\sin\theta
\]

cart\ (tangential)\ (centripetal)

\[
\sum F = ma \Rightarrow f(t) - kx - cx
\]

\[
= m(\ddot{x} + L\ddot{\theta}\cos\theta - L\dot{\theta}^2\sin\theta) + M\ddot{x}
\]

\[
\theta \rightarrow mL^2\ddot{\theta} + mL\dddot{x}\cos\theta + mgL\sin\theta = 0
\]

\[
x \rightarrow (M+m)\ddot{x} + m(L\ddot{\theta}\cos\theta - L\dot{\theta}^2\sin\theta) + cx + kx = f(t)
\]
Example - Cart-Pendulum Problem

For small motion, the equations can be linearized

\[ \cos \theta \approx 1; \sin \theta \approx \theta; \dot{\theta}^2 \approx 0 \]

\[ mL^2 \ddot{\theta} + mL \ddot{x} + mgL \theta = 0 \]

\[ (M+m) \ddot{x} + mL \ddot{\theta} + c \dot{x} + kx = f(t) \]

or in matrix forms as

\[
\begin{bmatrix}
    mL^2 & mL \\
    mL & (M+m)
\end{bmatrix}
\begin{bmatrix}
    \ddot{\theta} \\
    \ddot{x}
\end{bmatrix}
+ \begin{bmatrix}
    0 & 0 \\
    0 & c
\end{bmatrix}
\begin{bmatrix}
    \dot{\theta} \\
    \dot{x}
\end{bmatrix}
+ \begin{bmatrix}
    mgL & 0 \\
    0 & k
\end{bmatrix}
\begin{bmatrix}
    \theta \\
    x
\end{bmatrix}
= \begin{bmatrix}
    0 \\
    f(t)
\end{bmatrix}
\]
\[ \frac{k_{\text{eq}}}{k_1 + k_2} = \frac{k_1 k_2}{k_1 + k_2} \]

\[ \ln \frac{x_1}{x_2} \approx 2\pi\zeta \]

\[ F(s) \rightarrow \frac{1}{MS^2 + CS + K} \rightarrow x(s) \]