System Model Representation

\[ F(s) \xrightarrow{\frac{1}{MS^2 + CS + K}} x(s) \]
System Model Representation

System models can be developed in a variety of forms. Once the governing equations are developed, then the system characteristics and response can be determined.

Consider a simple mechanical system subjected to initial conditions

\[ x_0 \text{ and } \dot{x}_0 \]
System Model Representation

The governing differential equation (chap 4) describing the motion is

\[ m\ddot{x} + c\dot{x} + kx = 0 \quad \text{with} \quad x(0) = x_0; \quad \dot{x}(0) = \dot{x}_0 \]

Note that there is only one generalized coordinate to describe the motion - \( x \)

Note if a force is applied instead of an initial condition, there is still only one generalized equation

\[ m\ddot{x} + c\dot{x} + kx = f(t) \]
State-Space Representation

Another approach commonly used is to represent the system as a larger system of equations - The second order ODE with two variables to describe the “state” of the system rather than one variable.

The number of state variable depends on the number of possible initial conditions.

For the 2\textsuperscript{nd} order ODE

\[ m\ddot{x} + c\dot{x} + kx = f(t) \quad \text{IC} \quad x_0 \quad \& \quad \dot{x}_0 \]

has two state variables - displacement and velocity.
Example

The system is described by

\[ m\ddot{x} + c\dot{x} + kx = f(t) \quad \text{IC } x_0 \ & \dot{x}_0 \]

The two mutual conditions of \( x_0 \ & \dot{x}_0 \) are required to completely describe the “state” of the system. Hence there are two state variables.
Example - State Space

**Assume**

\[
\begin{align*}
\ddot{x}_1 + \dot{x}_1 + x_1 - x_2 &= 0 \\
\dot{x}_2 + 2x_1 + x_2 &= 0
\end{align*}
\]

In order to solve this, there are three mutual conditions required namely \( x_1(0), x_2(0), \dot{x}_1(0) \)
State Space - General Formulation

State-variable equations

\[
\begin{aligned}
\dot{x}_1 &= f_1(x_1, \ldots, x_n; u_1 \ldots u_m; t) \\
\dot{x}_2 &= f_2(x_1, \ldots, x_n; u_1 \ldots u_m; t) \\
&\vdots \\
\dot{x}_n &= f_n(x_1, \ldots, x_n; u_1 \ldots u_m; t)
\end{aligned}
\]

Where

- $x$ is a state variable
- $u$ is an input
- $f$ is some function
State Space - General Formulation

System Outputs

\[
\begin{align*}
    y_1 &= h(x_1, \ldots, x_n; u_1 \ldots u_m; t) \\
    y_2 &= h(x_1, \ldots, x_n; u_1 \ldots u_m; t) \\
    &\vdots
\end{align*}
\]

State variables can be expressed as

\[
\dot{x} = f(x, u, t)
\]

System variables can be expressed as

\[
y = h(x, u, t)
\]

Note that \( f \) and \( h \) can be general nonlinear functions
State Space - General Formulation

For a linear system, this simplifies to

\[
\begin{align*}
\dot{x}_1 &= a_{11}x_1 + \ldots + a_{1n}x_n + b_{11}u_1 + \ldots + b_{1m}u_m \\
\dot{x}_2 &= a_{21}x_1 + \ldots + a_{2n}x_n + b_{21}u_1 + \ldots + b_{2m}u_m \\
& \quad \vdots \\
\dot{x}_n &= a_{n1}x_1 + \ldots + a_{nn}x_n + b_{n1}u_1 + \ldots + b_{nm}u_m
\end{align*}
\]

\[
\begin{align*}
y_1 &= c_{11}x_1 + \ldots + c_{1n}x_n + d_{11}u_1 + \ldots + d_{1m}u_m \\
y_2 &= c_{21}x_1 + \ldots + c_{2n}x_n + d_{21}u_1 + \ldots + d_{2m}u_m \\
& \quad \vdots \\
y_n &= c_{p1}x_1 + \ldots + c_{pn}x_n + d_{p1}u_1 + \ldots + d_{pm}u_m
\end{align*}
\]
### State Space - General Formulation

**State Vector**

\[ x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \]

**Output Vector**

\[ y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix} \]

**Input Vector**

\[ u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} \]
**State Space - General Formulation**

**State Matrix**

\[
A = \begin{bmatrix}
  a_{11} & \rightarrow & a_{ln} \\
  \downarrow & & \downarrow \\
  a_{n1} & \rightarrow & a_{nn}
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
  b_{11} & \rightarrow & b_{1m} \\
  \downarrow & & \downarrow \\
  b_{m1} & \rightarrow & b_{mm}
\end{bmatrix}
\]

**Input Matrix**

**Output Matrix**

\[
C = \begin{bmatrix}
  c_{11} & \rightarrow & c_{ln} \\
  \downarrow & & \downarrow \\
  c_{n1} & \rightarrow & c_{nn}
\end{bmatrix}
\]

**Transmission Matrix**

\[
D = \begin{bmatrix}
  d_{11} & \rightarrow & d_{1m} \\
  \downarrow & & \downarrow \\
  d_{m1} & \rightarrow & d_{mm}
\end{bmatrix}
\]

**State Equation**

\[
\{\dot{x}\} = [A]\{x\} + [B]\{u\}
\]

**Output Equation**

\[
\{y\} = [C]\{x\} + [D]\{u\}
\]
Example - State Space

Governing Equation

\[ m\ddot{x} + c\dot{x} + kx = f(t) \]

The state variables are 

\[ x_1 = x \quad \& \quad x_2 = \dot{x} \]

In order to rewrite the governing equation as 2 first order ODE, the following relationship exists

\[ \dot{x}_1 = x_2 \]
Example - State Space

Substituting $\dot{x}_1 = x_2$ into the governing equation

$$m\ddot{x}_2 + c x_2 + k x_1 = f(t)$$

which is

$$\dot{x}_2 = \frac{1}{m} \left[ -c x_2 - k x_1 + f(t) \right]$$

Now the state variables equations are:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{1}{m} \left[ -c x_2 - k x_1 + f(t) \right]$$
Example - State Space

This can be written in matrix form as:

\[
\begin{align*}
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} &= \begin{bmatrix}
0 & 1 \\
-\frac{k}{m} & -\frac{c}{m}
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} + \begin{bmatrix}
0 \\
\frac{1}{m}
\end{bmatrix} f(t)
\end{align*}
\]

which is:

\[
y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 0 \cdot u
\]
Example - State Space

\[
\begin{align*}
\{\dot{x}\} &= [A]\{x\} + [B]\{u\} \\
\{\dot{x}_1\} &= \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}\{x\} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}f(t) \\
\{y\} &= [C]\{x\} + [D]\{u\} \\
y &= \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 0 \cdot u
\end{align*}
\]
**Example - Electrical Circuit**

**EXAMPLE 3.8.** Consider the electrical circuit in Fig. 3.6, in which the voltage $e(t)$ is the input, and $q_1$ and $q_2$ denote electric charges. The constant parameters $R$, $L_1$, $L_2$, $C_1$, and $C_2$ denote resistance, inductances, and capacitances, respectively. The system’s governing equations are obtained through application of Kirchhoff’s voltage law (see Chapter 5), as

\[
\begin{align*}
L_1 \ddot{q}_1 + R(\dot{q}_1 - \dot{q}_2) + \frac{1}{C_1} q_1 &= e \\
L_2 \ddot{q}_2 + R(\dot{q}_2 - \dot{q}_1) + \frac{1}{C_2} q_2 &= 0
\end{align*}
\]  
(3.20a)

(3.20b)

(a) By choosing a suitable set of state variables, obtain the state equation.

(b) Assuming that the system output is $q_1$, find the output equation. Repeat for the case in which the outputs are $q_1$ and $\dot{q}_1$.

*Source: Dynamic Systems - Vu & Esfandiari*
Example - Electrical Circuit

Solution

(a) Because each differential equation in Eq. (3.20) is second-order, a total of four initial conditions are required; hence, there exist four state variables. Therefore, state variables should be chosen as

\[
\begin{align*}
  x_1 &= q_1, & x_2 &= q_2, & x_3 &= \dot{q}_1, & x_4 &= \dot{q}_2
\end{align*}
\]

As a result, the state-variable equations are

\[
\begin{align*}
  \dot{x}_1 &= x_3 \\
  \dot{x}_2 &= x_4 \\
  \dot{x}_3 &= \dot{q}_1 = \frac{1}{L_1} \left[-R(x_3 - x_4) - \frac{1}{C_1} x_1 + e(t) \right] \\
  \dot{x}_4 &= \dot{q}_2 = \frac{1}{L_2} \left[-R(x_4 - x_3) - \frac{1}{C_2} x_2 \right]
\end{align*}
\]

Equations (3.21)

The third and fourth equations are obtained by substituting the state variables in Eqs. (3.20a) and (3.20b), and the first two are automatic relations be-

Source: Dynamic Systems - Vu & Esfandiari
Example - Electrical Circuit

tween the state variables, independent of the dynamics. The state equation is obtained by expressing Eq. (3.21) in matrix form, $\dot{x} = Ax + Bu$, as

$$\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-1/(L_1 C_1) & 0 & -R/L_1 & R/L_1 \\
0 & -1/(L_2 C_2) & R/L_2 & -R/L_2
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
1/L_1 \\
0
\end{bmatrix} e(t)
$$

(b) By assumption, the output is $q_1$, that is,

$$y = \begin{cases} q_1 \\ \text{first charge} \end{cases} \quad \implies \quad y = \begin{cases} x_1 \\ \text{first state} \end{cases}
$$

and the output equation is $y = Cx + Du$ where $C = [1 \ 0 \ 0 \ 0]$, and $D = 0$. However, for the case of two outputs, $q_1$ and $\dot{q}_1$, we have

$$y = \begin{bmatrix} q_1 \\ \dot{q}_1 \end{bmatrix} \quad \implies \quad y = \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} \quad \begin{cases} \text{first charge} \\ \text{and current} \end{cases} \quad \begin{cases} \text{first state} \\ \text{third state} \end{cases}
$$

and the output equation is $y = Cx + Du$, where

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad D = 0
$$

Source: Dynamic Systems - Vu & Esfandiari
Input-Output Equations

Another common form is the input-output equation. A single Diff Eq with time derivatives

\[ y^{(n)} + a_1 y^{(n-1)} + \ldots + a_{n-1} \dot{y} + a_n y \]

\[ = b_0 u^{(m)} + b_1 u^{(m-1)} + \ldots + b_{m-1} \dot{u} + b_m u \]

Note: \( y^{(n)} \) is the \( n^{th} \) derivative
\( a \) and \( b \) are constant coefficients
\( y \) is system output
\( u \) is system input

These equations are normally cumbersome and a Laplace Transform is used resulting in a set of algebraic equations.
Example - I/O Equation

Example 3.11. Obtain the input-output equation for the mechanical system of Example 3.4, where the input and output are \( f(t) \) and \( x_1(t) \), respectively.

Solution. From previous results, the equations of motion are

\[
\begin{align*}
    m_1 \ddot{x}_1 + b_1 \dot{x}_1 + k_1 x_1 - k_2 (x_2 - x_1) - b_2 (\dot{x}_2 - \dot{x}_1) &= 0 \\
    m_2 \ddot{x}_2 + b_2 (\dot{x}_2 - \dot{x}_1) + k_2 (x_2 - x_1) &= f(t)
\end{align*}
\]  

(3.8a)  

(3.8b)

The input-output equation must be a differential equation in terms of \( f(t) \), \( x_1(t) \), and their time derivatives. Taking the Laplace transform of Eqs. (3.8a and b) results in

\[
\begin{align*}
    m_1 s^2 X_1(s) + b_1 s X_1(s) + k_1 X_1(s) - k_2 [X_2(s) - X_1(s)] - b_2 [s X_2(s) - s X_1(s)] &= 0 \\
    m_2 s^2 X_2(s) + b_2 [s X_2(s) - s X_1(s)] + k_2 [X_2(s) - X_1(s)] &= F(s)
\end{align*}
\]  

(3.28)

Collect like terms in Eq. (3.28) to obtain

\[
\begin{align*}
    [m_1 s^2 + (b_1 + b_2) s + k_1 + k_2] X_1(s) - (b_2 s + k_2) X_2(s) &= 0 \\
    -(b_2 s + k_2) X_1(s) + (m_2 s^2 + b_2 s + k_2) X_2(s) &= F(s)
\end{align*}
\]

or, in matrix form,

\[
\begin{bmatrix}
    m_1 s^2 + (b_1 + b_2) s + k_1 + k_2 & -(b_2 s + k_2) \\
    -(b_2 s + k_2) & m_2 s^2 + b_2 s + k_2
\end{bmatrix}
\begin{bmatrix}
    X_1(s) \\
    X_2(s)
\end{bmatrix} = \begin{bmatrix}
    0 \\
    F(s)
\end{bmatrix}
\]

Source: Dynamic Systems - Vu & Esfandiari
Example – I/O Equation

Because the output is $x_1$, directly solve the above for $X_1(s)$ via Cramer’s rule, as follows

$$X_1(s) = \begin{vmatrix} 0 & -(b_2s + k_2) \\ F(s) & m_2s^2 + b_2s + k_2 \\ m_1s^2 + (b_1 + b_2)s + k_1 + k_2 & -(b_2s + k_2) \\ -(b_2s + k_2) & m_2s^2 + b_2s + k_2 \end{vmatrix}$$

$$\Rightarrow \quad X_1(s) = \frac{(b_2s + k_2)F(s)}{[m_1s^2 + (b_1 + b_2)s + k_1 + k_2](m_2s^2 + b_2s + k_2) - (b_2s + k_2)^2}$$

(3.29)

Algebraic manipulation of Eq. (3.29) yields

$$[m_1m_2s^4 + [m_1b_2 + m_2(b_1 + b_2)]s^3 + [m_1k_2 + b_1b_2 + (k_1 + k_2)m_2]s^2$$

$$+ (b_2k_1 + b_1k_2)s + k_1k_2]X_1(s) = (b_2s + k_2)F(s)$$

In time domain, representing the system’s output and input, this equation then reads

$$m_1m_2x_1^{(4)} + [m_1b_2 + m_2(b_1 + b_2)]\ddot{x}_1 + [m_1k_2 + b_1b_2 + (k_1 + k_2)m_2]\dot{x}_1$$

$$+ (b_2k_1 + b_1k_2)\ddot{x}_1 + k_1k_2x_1 = b_2\ddot{f} + k_2f$$

(3.30)

which is the system’s input-output equation. As expected, Eq. (3.30) is a differential equation relating input $f$, output $x_1$, and their derivatives, and is precisely in the general form of Eq. (3.27).

\textbf{Source: Dynamic Systems - Vu & Esfandiari}
**System Transfer Function**

*Once again, consider the linear time-invariant differential equation as:*

\[ y^{(n)} + a_1 y^{(n-1)} + \ldots + a_{n-1} \dot{y} + a_n y = b_0 u^{(m)} + b_1 u^{(m-1)} + \ldots + b_{m-1} \dot{u} + b_m u \]

*Assuming the initial conditions are zero, the Laplace Transform yields:*

\[ \left( s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n \right) y(s) = \left( b_0 s^m + b_1 s^{m-1} + \ldots + b_{m-1} s + b_m \right) U(s) \]
System Transfer Function

So that the system transfer function is

\[ G(s) = \frac{Y(s)}{U(s)} = \frac{b_0s^m + b_1s^{m-1} + \ldots + b_{m-1}s + b_m}{s^n + a_1s^{n-1} + \ldots + a_{n-1}s + a_n} \]

This is the system transfer function for a single input-single output (SISO). If there are other inputs and outputs to consider, then the problem becomes a multiple input-multiple output (MIMO)
Relationship between state space and system transfer function

\[\begin{align*}
\text{State space} & \quad \dot{x} = Ax + Bu \\
& \quad y = Cx + Du \\
\text{Transfer function} & \quad G(s) = \frac{Y(s)}{U(s)} \\
\text{Laplace Transform of State equation (w/IC = 0)} & \quad sX(s) = AX(s) + BU(s) \\
& \quad Y(s) = CX(s) + DU(s) \\
& \quad \downarrow \\
& \quad (sI - A)X(s) = BU(s) \\
& \quad Y(s) = CX(s) + DU(s)
\end{align*}\]
**State space and system transfer function**

\[
X(s) = (sI - A)^{-1}BU(s) \\
Y(s) = CX(s) + DU(s)
\]

**Then**

\[
Y(s) = C(sI - A)^{-1}BU(s) + DU(s) \\
= \left[C(sI - A)^{-1}B + D\right]U(s)
\]

\[\therefore G(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D\]
State space and system transfer function

Realizing that  
\[ (sI - A)^{-1} \Rightarrow \frac{\text{adj}(sI - A)}{\det(sI - A)} \]

So that  
\[ G(s) = \frac{\text{Cadj}(sI - A)B}{\det(sI - A)} + D \]
Example - State Space Representation

The SDOF state equation:

\[ \dot{x} = Ax + Bu \]
\[ y = Cx + Du \]

\[ x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \]

\[ A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \]

\[ B = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} \]

\[ C = \begin{bmatrix} 1 & 0 \end{bmatrix} \]

\[ D = 0 \]

\[ u = f(t) \]
Example - State Space Representation

Now
\[(sI - A) = \begin{bmatrix}
  s & -1 \\
  \frac{k}{m} & s + \frac{c}{m}
\end{bmatrix}\]

\[(sI - A)^{-1} = \frac{1}{s(s + \frac{c}{m}) + \frac{k}{m}} \begin{bmatrix}
  s & -1 \\
  \frac{k}{m} & s + \frac{c}{m}
\end{bmatrix}\]

Then
\[G(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{1}{s(s + \frac{c}{m}) + \frac{k}{m}} \begin{bmatrix}
  s & -1 \\
  \frac{k}{m} & s + \frac{c}{m}
\end{bmatrix} \begin{bmatrix} 0 \\
  \frac{1}{m}
\end{bmatrix}\]

\[= \frac{1}{s(s + \frac{c}{m}) + \frac{k}{m}} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix}
  \frac{1}{m} \\
  (\frac{1}{m})(s + \frac{c}{m})
\end{bmatrix}\]
Example - State Space Representation

\[
\begin{align*}
\frac{1}{s(s + \frac{c}{m}) + \frac{k}{m} m} \cdot \frac{1}{m} = \\
\Rightarrow G(s) = \frac{1}{ms^2 + cs + k}
\end{align*}
\]

which is the system transfer function. State space representation can also be obtained from the input-output equation.
Linearization

Many times the equations describing a system may be non-linear. Often, these equations can be “linearized” to determine an acceptable solution. The basic approach is to perform a Taylor Series Expansion about some “operating” point to find an equivalent linear set of equations with the assumption that the system will operate around this point.
Graphical Interpretation

The function can be evaluated in the vicinity of $\bar{x}$ and causes $\bar{f}$ to result.

Considering a small variation of $\Delta x$ results in $\Delta f$. In this region, the slope is approximated by

$$m = \frac{df}{dx} \bigg|_{x=\bar{x}}$$

Thus

$$f - \bar{f} = m(x - \bar{x}) \Rightarrow \Delta f = m\Delta x$$
Taylor Series Expansion

The analytical counterpart to the graphical representation is performed.

The Taylor Series Expansion about the point \( \bar{x} \) for a non-linear function \( f(x) \)

\[
f(x) = f(\bar{x}) + f'(x) \bigg|_{x=\bar{x}} (x - \bar{x}) + \frac{1}{2!} f''(x) \bigg|_{x=\bar{x}} (x - \bar{x})^2 + \ldots
\]

In the event that \((x - \bar{x})\) is small then higher order terms can be negligible. Thus

\[
f(x) \approx f(\bar{x}) + f'(x) \bigg|_{x=\bar{x}} (x - \bar{x})
\]

OR as done graphically:

\[
y \approx \bar{y} + f'(x) \bigg|_{x=\bar{x}} \Delta x
\]
Example - Linearization

**EXAMPLE 3.22.** Linearize the nonlinear function defined by \( y = x^2 \) about the operating point \((1, 1)\). Investigate the accuracy of the linearized model for \( x = 0.8 \) and \( x = 0.9 \).

**Solution.** By assumption, \( \bar{x} = 1 \) and \( \bar{y} = 1 \). The nonlinear function is a function of one variable, \( x \), and from Eq. (3.64), we have

\[
y \approx \bar{y} + f'(x)|_{x=\bar{x}} \Delta x \quad \Rightarrow \quad y \approx \bar{y} + [2x]_{x=1} \Delta x
\]

\[
\Rightarrow \quad y_{\text{approx}} \approx 1 + 2 \Delta x
\]

(3.65)

This describes the linear approximation of the original function in a small neighborhood of the operating point. Because the graph of \( y = x^2 \) is a smooth curve, at least in a close vicinity of \( P \), then the degree of accuracy of the linear model solely depends on how far \( x \) is located from \( P \). For instance, if \( x = 0.8 \), as in Fig. 3.11, then by Eq. (3.65), we have

\[
y_{\text{approx}} = 1 + 2 \Delta x = 1 + 2(0.8 - 1) = 0.60
\]

while

\[
y_{\text{actual}} = (0.8)^2 = 0.64
\]

which shows a 6 percent relative error. On the other hand, when \( x = 0.9 \), the numerical results are

\[
y_{\text{approx}} = 1 + 2 \Delta x = 1 + 2(0.9 - 1) = 0.80
\]

*Source: Dynamic Systems - Vu & Esfandiari*
Example - Linearization

FIGURE 3.11 Approximation of a nonlinear curve about the operating point.

while \( y_{\text{actual}} = (0.9)^2 = 0.81 \)

indicating that the relative error has been reduced to 1.1 percent.

Source: Dynamic Systems - Vu & Esfandiari
Example

Car Suspension

Mount Stiffness
Example - Nonlinear Spring

EXAMPLE 3.25. Consider the mechanical system shown in Fig. 3.13 involving a nonlinear spring, where \( f_s(x) = x|x| \) denotes the nonlinear spring force, as shown in Fig. 3.14. \( u(t) \) represents the applied force (input) and is assumed to be the unit-step function. The system is subjected to initial conditions \( x(0) = 0 \) and \( \dot{x}(0) = 1 \).

The equation of motion is given by

\[
\ddot{x} + \dot{x} + \frac{x|x|}{f_s(x)} = u(t)
\]

(3.69)

Source: Dynamic Systems - Vu & Esfandiari
Example - Nonlinear Spring

Determine an operating point and obtain a linear model approximating the nonlinear mechanical system about this point. Identify the initial conditions associated with the linear model.

Solution. Equation (3.69) clearly contains a nonlinear term, $x|x|$, which must be linearized in a small neighborhood of an operating point. The operating point, however, is not completely available and must be determined while the above-mentioned facts, labeled (1), (2), and (3), are taken into consideration. To this end, in Eq. (3.69), substitute $x = \bar{x}$ and $\bar{u} = \bar{u} = 1$, as the average value of the unit-step function is 1. Then, we have

$$\dot{\bar{x}} + \ddot{\bar{x}} + \bar{x}|\bar{x}| = 1 \implies \bar{x} | \bar{x} | = 1$$

(3.70)

Source: Dynamic Systems - Vu & Esfandiari
Example - Nonlinear Spring

Because of the presence of the absolute value, two cases must be considered as follows:

Case 1. Assume $\bar{x} > 0$. This implies that $|\bar{x}| = \bar{x}$, and Eq. (3.70) becomes $\bar{x}^2 = 1$. Solution yields

$$\bar{x} = \pm 1 \quad \implies \quad \bar{x} = 1$$

by assumption, $\bar{x} > 0$

Case 2. Assume $\bar{x} < 0$. This implies that $|\bar{x}| = -\bar{x}$, and Eq. (3.70) becomes $-\bar{x}^2 = 1$. This yields

$$\bar{x}^2 = -1$$

so no real solution exists. Therefore, Case 1 provides the only valid solution, $\bar{x} = 1$. Consider the nonlinear term $f_3(x) = x|x|$. Its complete analytical description, and its derivative with respect to $x$, can be given as

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Example - Nonlinear Spring

\[ f_s(x) = x|x| = \begin{cases} \frac{x^2}{2} & \text{if } x \geq 0 \\ -\frac{x^2}{2} & \text{if } x < 0 \end{cases} \Rightarrow f_s'(x) = \begin{cases} 2x & \text{if } x \geq 0 \\ -2x & \text{if } x < 0 \end{cases} \quad (3.71) \]

\( f_s(x) \) is linearized about the operating point, via Eq. (3.64), as follows:

\[ f_s(x) = f_s(x) + f_s'(x) \bigg|_{x=x} \Delta x \quad (3.72) \]

Because \( x = 1 > 0 \), the first segments in the descriptions of \( f_s(x) \) and \( f_s'(x) \) in Eq. (3.71) will apply. Hence, we have \( f_s(1) = 1 \) and \( f_s'(1) = [2x]_{x=1} = 2 \). The latter is, of course, the slope of the tangent line to the graph of \( f_s(x) \) at \( (1, 1) \), illustrated in Fig. 3.14. Therefore, Eq. (3.72) reduces to

\[ f_s(x) \approx 1 + 2 \Delta x \quad (3.73) \]

which is clearly a linear expression. This completes the linearization process of the only nonlinear term in the equation of motion.

Next, in order to obtain a linear model that approximates the original nonlinear system, the following steps need to be undertaken:

- Recall that \( \Delta x = x - \bar{x} \) so that \( x = \bar{x} + \Delta x = 1 + \Delta x \). Substitute for \( x \) in the original nonlinear model, Eq. (3.69), taking into account that \( \bar{x} \) is a constant; i.e., \( \dot{\bar{x}} = \ddot{x} = 0 \).
- In Eq. (3.69), substitute \( u(t) = \bar{u} + \Delta u(t) = 1 + \Delta u(t) \) with \( \Delta u(t) = 0 \); because \( u \) is the unit-step and \( \bar{u} = 1 \), there are zero deviations. Also replace the nonlinear term by its linear approximation, given by Eq. (3.73), to obtain

\[
\frac{d^2}{dt^2}(1 + \Delta x) + \frac{d}{dt}(1 + \Delta x) + \left(1 + 2 \Delta x\right) \left.\right|_{\text{linear approximation}} = 1 + \Delta u(t) \]
Example – Nonlinear Spring

Simplification yields

\[ \Delta x + \dot{\Delta} x + 1 + 2 \Delta x = 1 \implies \Delta x + \dot{\Delta} x + 2 \Delta x = 0 \quad (3.74) \]

Equation (3.74), a linear second-order differential equation in \( \Delta x \) with constant coefficients, is a linear model that approximates the original system. Notice that the new variable for the linear model is the increment variable \( \Delta x \), in place of \( x \). The input to the linear model is the increment variable corresponding to \( u(t) \), which is zero in this case.

- Finally, to obtain the complete description of the linear model, a new set of initial conditions should be specified in relation to \( \Delta x \). To this end, recall that \( \Delta x(t) = x(t) - \bar{x} \). Evaluation at \( t = 0 \) results in

\[ \Delta x(0) = x(0) - \bar{x} = -1 \]

and differentiation yields

\[ \dot{\Delta} x(t) = \dot{x}(t) - \dot{\bar{x}} = \dot{x}(t) \quad \text{at } t = 0 \implies \Delta x(0) = \dot{x}(0) = 1 \]

Hence, the linear model is completely described by

\[ \Delta x + \dot{\Delta} x + 2 \Delta x = 0, \quad \Delta x(0) = -1, \quad \dot{\Delta} x(0) = 1 \]

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For convenience, we may use \( \ddot{x} \) in place of \( \Delta x \) so that the linear model is rewritten as

\[
\dddot{x} + \ddot{x} + 2\dot{x} = 0, \quad \dot{x}(0) = -1, \quad \ddot{x}(0) = 1
\]

Note that because \( \ddot{x} \) denotes deviations of \( x \) relative to the operating point, (i.e., \( x = \ddot{x} + \dot{x} \)), the solution to the linearized model does not approximate the solution to the nonlinear system.

Procedure: The steps undertaken in Example 3.25 suggest a general procedure that must be followed. This may be outlined as a four-step procedure:

1. To determine the operating point, replace the dependent variable(s) such as \( x \) and \( y \) by \( \ddot{x} \) and \( \ddot{y} \), and the input \( u \) by its average value \( \bar{u} \), if specified. Solve the resulting (nonlinear) algebraic equation(s) for constants such as \( \ddot{x} \) and \( \ddot{y} \).
   * In the event that the input is composed of two or more functions, set the time-varying portion(s) equal to zero. For instance, if \( u(t) = k + g(t) \), with \( k \) constant, set \( g(t) = 0 \) so that \( \bar{u} = k \).
2. Linearize the nonlinear term(s) about the operating point via Taylor series expansion.
3. In the original nonlinear model, replace the dependent variable(s) such as \( x \) and \( y \) by \( \ddot{x} + \Delta x \) and \( \ddot{y} + \Delta y \), the input \( u \) by \( \bar{u} + \Delta u \), and the nonlinear term(s) by their linear approximations.
4. Properly adjust the initial conditions.

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