System Response

Transfer Function

\[ Y(S) = H(S) \times F(S) \]

Response Models

\[ y(t) = h(t) \otimes f(t) \]

Frequency Domain

\[ y(\omega) = H(\omega) \times F(\omega) \]

Laplace Domain

\[ S(f) \rightarrow \frac{1}{MS^2 + CS + K} \rightarrow x(s) \]

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Transient Response - First-order System

\[ \tau \dot{y} + y = f(t) \quad \tau > 0 \quad \text{I.C. } y(0) = y_0 \]

Free Response
No forcing function - Only I.C.
Laplace Transform
\[ \tau [sY(s) - y_0] + Y(s) = 0 \implies Y(s) = \frac{\tau y_0}{\tau s + 1} = \frac{y_0}{s + \frac{1}{\tau}} \]

Inverse Laplace
\[ y(t) = y_0 e^{-\left(\frac{t}{\tau}\right)} \]

start at \( y_0 \) and exponentially decay to zero as \( t \to \infty \)

Source: Dynamic Systems - Vu & Esfandiari
Transient Response - First-order System

Forced Response - Step Response

\[ f(t) = A \quad \text{if } t > 0 \quad \text{otherwise} = 0 \]

Laplace Transform

\[ \tau [sY(s) - y_o] + Y(s) = \frac{A}{s} \quad \Rightarrow \quad Y(s) = \frac{\tau y_o s + A}{s(\tau s + 1)} \]

break up using partial fraction expansion

\[ Y(s) = \frac{y_o s + \frac{A}{\tau}}{s \left( s + \frac{1}{\tau} \right)} = \frac{c_1}{s} + \frac{c_2}{s + \frac{1}{\tau}} \]
Transient Response - First-order System

where

\[ c_1 = sY(s) \bigg|_{s=0} = \frac{\tau y_o s + A}{s + 1} \bigg|_{s=0} = A \]

\[ c_2 = \left(s + \frac{1}{\tau}\right)Y(s) \bigg|_{s=-1/\tau} = \frac{y_o s + \frac{A}{s}}{s} \bigg|_{s=-1/\tau} = y_o - A \]

\[ \therefore Y(s) = \frac{A}{s} + \left(y_o - A\right)\frac{1}{s + \frac{1}{\tau}} \]
Transient Response - First-order System

Inverse Laplace

\[ y_{STEP}(t) = A + (y_o - A)e^{-t/\tau} \]

if \( y_o = 0 \) → \( y_{STEP}(t) = A\left(1 - e^{-t/\tau}\right) \)

\( y(\tau) = 0.632A \)
\( y(2\tau) = 0.865A \)
\( y(3\tau) = 0.950A \)
\( y(4\tau) = 0.982A \)
\( y(5\tau) = 0.995A \)

Source: Dynamic Systems - Vu & Esfandiari
Transient Response - First-order System

Forced Response - Ramp Response

\[ f(t) = At \quad \text{if } t > 0 \quad \text{otherwise } = 0 \]

Laplace Transform

\[
\tau [sY(s) - y_o] + Y(s) = \frac{A}{s^2}
\]

\[
Y(s) = \frac{A}{\tau} + \frac{y_o s^2}{s^2 \left( s + \frac{1}{\tau} \right)} = \frac{B_2}{s^2} + \frac{B_1}{s} + \frac{C}{s + \frac{1}{\tau}}
\]

where \( B_2 = A; \quad B_1 = -\tau A; \quad C = y_o + \tau A \)
Transient Response - First-order System

Inverse Laplace

\[ y_{\text{RAMP}}(t) = At - A\tau + A\tau e^{-\frac{t}{\tau}} \]

Note that the SS response is \( At - A\tau \)
After the transient portion decays, the response will track the ramp but with an error of \( A\tau \).

Source: Dynamic Systems - Vu & Esfandiari
**Transient Response – Second-order System**

\[ \ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = f(t) \quad \text{I.C. } x_0; \dot{x}_0 \]

**Free Response – No forcing function – Only I.C.**

**Laplace Transform**

\[
\left[ s^2X(s) - sx(0) - \dot{x}(0) \right] + 2\zeta\omega_n \left[ sX(s) - x(0) \right] + \omega_n^2 X(s) = 0 \]

\[
X(s) = \frac{(s + 2\zeta\omega_n)x_0 - x_0}{s^2 + 2\zeta\omega_n s + \omega_n^2}
\]

**Solution form depends on the poles of the characteristic equation**

\[
s^2 + 2\zeta\omega_n s + \omega_n^2 = 0
\]

\[
\therefore s_{1,2} = -\zeta\omega_n \pm \sqrt{(\zeta\omega_n)^2 - \omega_n^2}
\]

\[
= -\zeta\omega_n \pm \omega_n \sqrt{\zeta^2 - 1}
\]
Transient Response - Second-order System

Three cases of damping

• Case 1 – Overdamped (\(\zeta > 1\))

• Case 2 – Critically damped (\(\zeta = 1\))

• Case 3 – Underdamped (\(0 < \zeta < 1\))

• Case 4 – Undamped (\(\zeta = 0\))
**Transient Response - Second-order System**

**Case 1 - Overdamped (\(\zeta>1\)) - Two Real Roots**

\[
X(s) = \frac{(s + 2\zeta\omega_n)x_0 + v_0}{s^2 + 2\zeta\omega_ns + \omega_n^2} = \frac{(s + 2\zeta\omega_n)x_0 + v_0}{(s - s_1)(s - s_2)}
\]

\[
s_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}; \quad s_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}
\]

**Partial Fraction Expansion**

\[
X(s) = \frac{A_1}{s - s_1} + \frac{A_2}{s - s_2}
\]

\[
A_1 = \frac{\omega_n(\zeta + \sqrt{\zeta^2 - 1})x_0 + v_0}{2\omega_n\sqrt{\zeta^2 - 1}}
\]

\[
A_2 = \frac{-\omega_n(\zeta - \sqrt{\zeta^2 - 1})x_0 + v_0}{2\omega_n\sqrt{\zeta^2 - 1}}
\]
Transmit Response - Second-order System

Case 1 - Overdamped ($\zeta > 1$) - Two Real Roots

\[
x(t) = \left[ \frac{\omega_n \left( \zeta + \sqrt{\zeta^2 - 1} \right) x_0 + v_0}{2\omega_n \sqrt{\zeta^2 - 1}} \right] e^{-\omega_n (\zeta - \sqrt{\zeta^2 - 1}) t} \\
- \left[ \frac{\omega_n \left( \zeta - \sqrt{\zeta^2 - 1} \right) x_0 + v_0}{2\omega_n \sqrt{\zeta^2 - 1}} \right] e^{\omega_n (\zeta + \sqrt{\zeta^2 - 1}) t}
\]
Transient Response - Second-order System

Case 2 - Critically damped ($\zeta=1$) - Two Real Repeated Roots

$$s_{1,2} = -\omega_n \text{ (two real repeated roots)}$$

$$X(s) = \frac{(s + 2\omega_n)x_0 + v_0}{s^2 + 2\omega_n s + \omega_n^2} = \frac{(s + \omega_n)x_0 + \omega_n x_0 + v_0}{(s + \omega_n)^2}$$

$$X(s) = \frac{x_0}{s + \omega_n} + \frac{\omega_n x_0 + v_0}{(s + \omega_n)^2}$$

Inverse Laplace

$$x(t) = x_0 e^{-\omega_n t} + (\omega_n x_0 + v_0)te^{-\omega_n t}$$
Transient Response - Second-order System

Case 3 - Underdamped (0<\(\zeta\)<1) - Complex Conjugate Pair

Define:

\[
\sigma = \zeta \omega_n \\
\omega_d = \omega_n \sqrt{1 - \zeta^2}
\]

Then

\[
s^2 + 2\zeta\omega_n s + \omega_n^2 = (s + \zeta\omega_n)^2 - \zeta^2 \omega_n^2 + \omega_n^2 = (s + \sigma)^2 + \omega_d^2
\]

Then

\[
X(s) = \frac{(s + 2\zeta\omega_n)x_0 + v_0}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{(s + 2\sigma)x_0 + x_0 + v_0}{(s + \sigma)^2 + \omega_d^2}
\]
Transient Response - Second-order System

which can be rearranged into

\[ X(s) = \frac{(s + \sigma)x_0}{(s + \sigma)^2 + \omega_d^2} + \frac{\sigma x_0 + v_0}{(s + \sigma)^2 + \omega_d^2} \]

\[ = \frac{(s + \sigma)x_0}{(s + \sigma)^2 + \omega_d^2} + \frac{\sigma x_0 + v_0}{\omega_d \left( (s + \sigma)^2 + \omega_d^2 \right)} \]

Inverse Laplace

\[ x(t) = x_0 e^{-\sigma t} \cos \omega_d t + \frac{\sigma x_0 + v_0}{\omega_d} e^{-\sigma t} \sin \omega_d t \]

OR

\[ = e^{-\sigma t} \left[ x_0 \cos \omega_d t + \frac{\sigma x_0 + v_0}{\omega_d} \sin \omega_d t \right] \]
Forced Response - Unit Impulse

\[ \ddot{x} + 2\zeta \omega_n \dot{x} + \omega_n^2 x = f(t) = \delta(t) \quad \text{with I.C.} \quad x_0; \dot{x}_0 \]

Laplace Transform

\[
\left[ s^2 X(s) - sx(0) - \dot{x}(0) \right] + 2\zeta \omega_n \left[ sX(s) - x(0) \right] + \omega_n^2 X(s) = F(s)
\]

Grouping terms

\[
X(s) = \frac{F(s)}{s^2 + 2\zeta \omega_n s + \omega_n^2} + \frac{(s + 2\zeta \omega_n)x_0 + v_0}{s^2 + 2\zeta \omega_n s + \omega_n^2}
\]

If I.C. are zero, then the result is the system transfer function

\[
H(s) = \frac{X(s)}{F(s)} = \frac{1}{s^2 + 2\zeta \omega_n s + \omega_n^2}
\]

The inverse Laplace of this yields the impulse response of the system. (Note that I.C. are zero)
Forced Response - Unit Impulse

Case 1 - Overdamped

\[ x(t) = \frac{1}{2\omega_n \sqrt{\zeta^2 - 1}} e^{-\omega_n (\zeta - \sqrt{\zeta^2 - 1}) t} \]  

\[ - \frac{1}{2\omega_n \sqrt{\zeta^2 - 1}} e^{-\omega_n (\zeta + \sqrt{\zeta^2 - 1}) t} \]  

\[ + \left[ \frac{\omega_n \left( \zeta + \sqrt{\zeta^2 - 1} \right) x_0 + v_0}{2\omega_n \sqrt{\zeta^2 - 1}} \right] e^{-\omega_n (\zeta - \sqrt{\zeta^2 - 1}) t} \]  

\[ - \left[ \frac{\omega_n \left( \zeta - \sqrt{\zeta^2 - 1} \right) x_0 + v_0}{2\omega_n \sqrt{\zeta^2 - 1}} \right] e^{-\omega_n (\zeta + \sqrt{\zeta^2 - 1}) t} \]  

I.C.
Forced Response - Unit Impulse

Case 2 - Critically Damped

\[ x(t) = te^{-\omega_n t} + x_0 e^{-\omega_n t} + (\omega_n x_0 + v_0)te^{-\omega_n t} \]

I.C.
Forced Response - Unit Impulse

Case 3 - Underdamped

\[ x(t) = \frac{1}{\omega_d} e^{-\sigma t} \sin \omega_d t + e^{-\sigma t} \left[ x_o \cos \omega_d t + \frac{(\sigma x_o + v_o)}{\omega_d} \sin \omega_d t \right] \]

I.C.

Case 4 - Undamped

\[ x(t) = \frac{1}{\omega_n} \sin \omega_n t + x_o \cos \omega_n t + \frac{v_o}{\omega_n} \sin \omega_n t \]
Forced Response - Unit Impulse

FIGURE 7.7  Response of overdamped, critically damped, underdamped, and undamped second-order systems to a unit impulse; $x_0 \neq 0, v_0 = 0$.

Source: Dynamic Systems - Vu & Esfandiari
Forced Response - Unit Impulse

FIGURE 7.8 Unit-impulse response of overdamped, critically damped, underdamped, and undamped second-order systems; zero initial conditions.

Source: Dynamic Systems - Vu & Esfandiari
Forced Response - Unit Step

\[ \ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = f(t) = u(t) \quad \text{with I.C.} \quad x_0; \dot{x}_0 \]

**Case 1. Overdamped, \( \zeta > 1 \)**

\[
x(t) = \frac{1}{\omega_n^2} \left( \frac{1}{1 + \frac{1}{2\sqrt{\xi^2 - 1}}} \left[ \frac{1}{\zeta + \sqrt{\xi^2 - 1}} e^{-\omega_n(\xi - \sqrt{\xi^2 - 1})t} \right.ight.
\]
\[
+ \left. \left. \frac{1}{\xi + \sqrt{\xi^2 - 1}} e^{-\omega_n(\xi + \sqrt{\xi^2 - 1})t} \right] \right) + \frac{\omega_n(\xi + \sqrt{\xi^2 - 1})x_0 + v_0}{2\omega_n \sqrt{\xi^2 - 1}} e^{-\omega_n(\xi - \sqrt{\xi^2 - 1})t} \]
\]

Response to unit step

\[
+ \left[ \frac{\omega_n(\xi + \sqrt{\xi^2 - 1})x_0 + v_0}{2\omega_n \sqrt{\xi^2 - 1}} \right] e^{-\omega_n(\xi - \sqrt{\xi^2 - 1})t} \]
\]

Response to initial conditions

**Case 2. Critically damped, \( \zeta = 1 \)**

\[
x(t) = \frac{1}{\omega_n^2} \left[ 1 - e^{-\omega_n(1 - \omega_n)t} \right] + x_0 e^{-\omega_n t} + (\omega_n x_0 + v_0) t e^{-\omega_n t} \]

Response to unit step \quad Response to initial conditions

Source: Dynamic Systems - Vu & Esfandiari
Forced Response - Unit Step

Case 3. Underdamped, $0 < \zeta < 1$

$$x(t) = \frac{1}{\omega_n^2} \left\{ 1 - e^{-\alpha t} \left[ \cos \omega_d t + \frac{\zeta}{\sqrt{1 - \zeta^2}} \sin \omega_d t \right] \right\}$$

Response to unit step

$$+ e^{-\alpha t} \left[ x_0 \cos \omega_d t + \left( \frac{\sigma x_0 + v_0}{\omega_d} \right) \sin \omega_d t \right]$$

Response to initial conditions

(7.35)

An alternative, and perhaps more convenient, form of Eq. (7.35) may be obtained as

$$x(t) = \frac{1}{\omega_n^2} \left\{ 1 - e^{-\alpha t} \frac{1}{\sqrt{1 - \zeta^2}} \sin(\omega_d t + \phi) \right\}$$

Response to unit step

$$+ e^{-\alpha t} \frac{\sqrt{(\sigma x_0 + v_0)^2 + \omega_d^2 v_0^2}}{\omega_d} \sin(\omega_d t + \psi)$$

Response to initial conditions

(7.36)

where

$$\phi = \tan^{-1} \frac{\sqrt{1 - \zeta^2}}{\zeta} \quad \text{and} \quad \psi = \tan^{-1} \frac{x_0 \omega_d}{\sigma x_0 + v_0}$$

Case 4. Undamped, $\zeta = 0$

$$x(t) = \frac{1}{\omega_n^2} (1 - \cos \omega_n t) + x_0 \cos \omega_n t + \frac{v_0}{\omega_n} \sin \omega_n t$$

Response to unit step  Response to initial conditions

(7.37)

Source: Dynamic Systems - Vu & Esfandiari
Forced Response - Unit Step

**FIGURE 7.10** Response of overdamped, critically damped, underdamped, and undamped second-order systems to a unit step; $x_0 \neq 0, v_0 = 0$.

*Source: Dynamic Systems - Vu & Esfandiari*
Forced Response – Unit Step

FIGURE 7.11 Unit-step response of overdamped, critically damped, underdamped, and undamped second-order systems; zero initial conditions.

Source: Dynamic Systems - Vu & Esfandiari
Frequency Response Function

The FRF can be obtained from the Fourier Transform of Input-Output Time Response (and is commonly done so in practice).

The FRF can also be obtained from the evaluation of the system transfer function at \( s = j\omega \).

For a 1\textsuperscript{st} order system

\[
H(s) = \frac{1}{\tau s + 1} \quad \Rightarrow \quad H(s)\bigg|_{s=j\omega} = H(j\omega) = \frac{1}{1 + j\omega\tau}
\]

For a 2\textsuperscript{nd} order system

\[
H(s) = \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2}
\]

\[
\Rightarrow \quad H(s)\bigg|_{s=j\omega} = H(j\omega) = \frac{1}{-\omega^2 + 2\zeta\omega_n j\omega + \omega_n^2}
\]
Frequency Response Function

The FRF for a mechanical system

\[
H(s) = \frac{1}{ms^2 + cs + k} \Rightarrow H(j\omega) = \frac{1}{-m\omega^2 + jce\omega + k}
\]

This is normally presented in a LOG-MAG and PHASE plot called a BODE DIAGRAM.
FRF - Bode Diagram - 1st Order

Source: Dynamic Systems - Vu & Esfandiari
FRF – Bode Diagram – 2nd Order

Source: Dynamic Systems – Vu & Esfandiari
Fourier Series

A periodic function is a function that satisfies the relationship

\[ f(t) = f(t + T) \]

where \( T \) is the period of the function

The fundamental angular frequency is defined by

\[ \omega_0 = \frac{2\pi}{T} \text{ (rad/sec)} \]
Fourier Series

Consider a family of waveforms

\[ f_1(t) = 1 \]
\[ f_2(t) = \sin(2\pi t) \]
\[ f_3(t) = \frac{1}{3}\sin(6\pi t) \]
\[ f_4(t) = \frac{1}{5}\sin(10\pi t) \]
\[ f_4(t) = \frac{1}{7}\sin(14\pi t) \]

Summed as

\[ g(t) = \sum_{n=1}^{N} f_n(t) \quad 1 < N \leq 5 \]
Fourier Series

The summed waveforms can be plotted in Matlab to approximate a square wave

```matlab
clear all

% Define the time array
t = 0:0.001:2;

% Individual waveforms
f1 = 1;
f2 = sin(2*pi*t);
f3 = (1/3)*sin(6*pi*t);
f4 = (1/5)*sin(10*pi*t);
f5 = (1/7)*sin(14*pi*t);

% Summation of the wave forms
f = f1+f2+f3+f4+f5

% Plot the waveforms as one, summed signal
figure(1)
plot(t,f)

% Plot the waveforms as individual signals
figure(2)
plot(t,f1,t,f2,t,f3,t,f4,t,f5)
```
Fourier Series

A Fourier Series representation of a real periodic function \( f(t) \) is based upon the summation of harmonically related sinusoidal components.

If the period is \( T \), then the harmonics are sinusoids with frequencies that are integer multiples of \( \omega_0 \); This is written as

\[
f_n(t) = a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)
\]

\[
= A_n \sin(n\omega_0 t + \phi_n)
\]

Note: These two are equivalent - one is written with sine and cosine terms - the other is written as a sin with amplitude and phase.
Fourier Series

The Fourier Series representation of an arbitrary periodic waveform $f(t)$ is an infinite sum of harmonically related sinusoids and is commonly written as

$$f(t) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left[ a_n \cos(n\omega_o t) + b_n \sin(n\omega_o t) \right]$$

or

$$f(t) = \frac{a_n}{2} \left( e^{jn\omega_o t} + e^{-jn\omega_o t} \right) + \frac{b_n}{2j} \left( e^{jn\omega_o t} + e^{-jn\omega_o t} \right)$$

$$= \frac{1}{2} \left( a_n - jb_n \right) e^{jn\omega_o t} + \frac{1}{2} \left( a_n + jb_n \right) e^{-jn\omega_o t}$$

$$= \sum_{n=-\infty}^{\infty} \left[ F_n e^{jn\omega_o t} \right]$$
Fourier Coefficients

The derivation of the expression for computing the coefficients in a Fourier Series is beyond the scope of this course. Without proof, the coefficients \( F_n \) are

\[
F_n = \frac{1}{T} \int_{t_1}^{t_1+T} f(t) e^{-jn\omega_0 t} dt
\]

which is evaluated over any periodic segment or in sinusoidal form

\[
a_n = \frac{1}{T} \int_{t_1}^{t_1+T} f(t) \cos(n\omega_0 t) dt
\]

\[
b_n = \frac{1}{T} \int_{t_1}^{t_1+T} f(t) \sin(n\omega_0 t) dt
\]
Response of Linear System Due to Periodic Inputs

Consider a linear SDOF system with a steady state periodic input (assume that the initial conditions and transients have decayed to zero)

The input excitation can be described by a Fourier Series representation as

\[ f(t) = \sum_{n=-\infty}^{\infty} F_n e^{j \omega_o t} \]

\[ = \frac{1}{2} a_o + \sum_{n=1}^{\infty} A_n \sin(n \omega_o t + \phi_n) \]
Response of Linear System Due to Periodic Inputs

The system output response is related to the input excitation through the system frequency response function as

\[ Y(j\omega) = H(j\omega) \cdot F(j\omega) \]

The \( n^{th} \) real harmonic input component of the Fourier Series

\[ f_n(t) = A_n \sin(n\omega_o t + \phi) \]

Generates an output sinusoidal component \( y_n(t) \) with a magnitude and phase determined by the \( n^{th} \) component of \( H(j\omega) \)

\[ y_n(t) = |H(jn\omega_o)|A_n \sin[n\omega_o + \phi_n + \angle H(jn\omega_o)] \]
Response of Linear System Due to Periodic Inputs

From superposition, the total output $y_n(t)$ is the sum of all the $n$ components as

$$y(t) = \sum_{n=0}^{\infty} y_n(t)$$

$$= \frac{1}{2} a_o H(j o) + \sum_{n=1}^{\infty} A_n |H(j n \omega_o)| \sin[n \omega_o t + \phi_n + \angle H(j n \omega_o)]$$

Note: This is also a Fourier Series with the same fundamental and harmonic frequencies as the input.

In complex form, this is written as

$$y_n(t) = H(j n \omega_o) F_n e^{j n \omega_o t}$$

The complete output Fourier Series is

$$y(t) = \sum_{n=-\infty}^{\infty} y_n(t) = \sum_{n=-\infty}^{\infty} H(j n \omega_o) F_n e^{j n \omega_o t}$$
Response of Linear System Due to Periodic Inputs

Consider a simple mass spring dashpot system that is under steady state sinusoidal excitation (but not a $\omega_n$ of a SDOF system)

**Time Domain**

$$f(t) = \sin(\omega t)$$

**Frequency Domain**

$$f(j\omega) \xrightarrow{\text{FFT}} h(j\omega) \xrightarrow{\text{System}} y(j\omega) \xrightarrow{\text{IFT}} y(t) = A \sin(\omega t + \phi)$$
Response of Linear System Due to Periodic Inputs

Now consider excitation at the natural frequency, $\omega_n$, of the SDOF system.
Response of Linear System Due to Periodic Inputs

Now consider a periodic input which is not sinusoidal. However, the Fourier Series representation is nothing more than the sum of different amplitude sine waves.
Response of Linear System Due to Periodic Inputs

Example 15.7

The first-order electric network shown in Fig. 15.7 is excited with the sawtooth function. Find an expression for the series representing the output \( V_{out}(t) \).
Response of Linear System Due to Periodic Inputs

Example 15.7 (cont.)

The electric network has a transfer function

\[ H(s) = \frac{1}{\text{RC}s + 1} \]

and therefore has a frequency response function

\[ |H(j\omega)| = \frac{1}{\sqrt{(\omega\text{RC})^2 + 1}} \]
\[ \angle H(j\omega) = \tan^{-1}(-\omega\text{RC}) \]

the input function \( u(t) \) may be represented by the Fourier Series

\[ u(t) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\omega_0 t) \]
Response of Linear System Due to Periodic Inputs

Example 15.7 (cont.)

At the output the series representation is

\[ y(t) = \sum_{n=1}^{\infty} H(j\omega) \left( \frac{2(-1)^{n+1}}{n\pi} \right) \sin\left[n\omega_0 t + \angle H(jn\omega_0)\right] \]

As an example consider the response if the period of the input is chosen to be \( T = \pi RC \) so that \( \omega_0 = \frac{2}{RC} \), then

\[ y(t) = \sum_{n=1}^{\infty} \left( \frac{2(-1)^{n+1}}{n\pi \sqrt{(n\omega_0 RC)^2 + 1}} \right) \sin\left[n\omega_0 t + \tan^{-1}(\frac{2n}{n\omega_0 RC})\right] \]
Response of Linear System Due to Periodic Inputs

Example 15.7 (cont.)

Figure 15.8 shows the computer response found by summing the first 100 terms of the Fourier Series.

![Graph showing the response of a first-order electric system to a sawtooth input.](image-url)
Response of Linear System Due to Periodic Inputs

Example 15.8

A cart shown in Figure 15.9a, with a mass \( m = 1.0 \, \text{kg} \) is supported on low friction bearings that exhibit a viscous drag \( B = 0.2 \, \text{N-s/m} \) and is coupled through a spring with stiffness \( K=25 \, \text{N/m} \) to a velocity source with a magnitude of 10 m/s but which switches direction every \( \pi \) seconds as shown in Figure 15.9b.

\[
V_{\text{in}}(t) = \begin{cases} 
10 \, \text{m/s} & 0 \leq t < \pi \\
-10 \, \text{m/s} & \pi \leq t < 2\pi 
\end{cases}
\]

The task is to find the resulting velocity of the mass \( v_m(t) \)
Response of Linear System Due to Periodic Inputs

Example 15.8 (cont.)

The input $\Omega(t)$ has a period of $T = 2\pi$ and a fundamental frequency of $\omega_0 = 2\pi/T = 1$ rad/s. The Fourier Series for the input contains only odd harmonics:

$$u(t) = \frac{20}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)} \sin[(2n-1)\omega_0 t]$$

$$= \frac{20}{\pi} \left[ \sin(\omega_0 t) + \frac{1}{3} \sin(3\omega_0 t) + \frac{1}{5} \sin(5\omega_0 t) + \ldots \right]$$
Response of Linear System Due to Periodic Inputs

Example 15.8 (cont.)

The frequency response of the system is

\[
H(j\omega) = \frac{25}{(25 - \omega^2) + j0.2\omega}
\]

which when evaluated at the harmonic frequencies of the input

\[n\omega_0 = n \text{ radians/s} \quad \text{is}
\]

\[
H(jn\omega_0) = \frac{25}{[(25 - n^2) + j0.2n]}
\]
### Response of Linear System Due to Periodic Inputs

**Example 15.8 (cont.)**

The following table summarizes the first five odd spectral components at the system input and output.

| $n\omega_0$ | $u_n$      | $|H(jn\omega_0)|$ | $\angle H(jn\omega_0)$ | $\gamma_n$          |
|-------------|------------|--------------------|------------------------|---------------------|
| 1           | $6.366 \sin(t)$ | 1.041              | $-0.008$               | $6.631 \sin(t - 0.008)$ |
| 3           | $2.122 \sin(3t)$ | 1.561              | $-0.038$               | $3.313 \sin(t - 0.038)$ |
| 5           | $1.273 \sin(5t)$ | 25.00              | $-1.571$               | $31.83 \sin(t - 1.571)$ |
| 7           | $0.909 \sin(7t)$ | 1.039              | $-3.083$               | $0.945 \sin(t - 3.083)$ |
| 9           | $0.707 \sin(9t)$ | 0.446              | $-3.109$               | $0.315 \sin(t - 3.109)$ |
Response of Linear System Due to Periodic Inputs

Example 15.8 (cont.)

Figure 15-10a shows the computed frequency response magnitude for the system and the relative gains and phase shifts (rad) associated with the first five terms in the series.