Fluid and Thermal Systems

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Fluid and Thermal Systems

Fluid and thermal systems are generally nonlinear. However, these systems can be linearized by evaluating them about some normal operating point.

Fluid systems are linearized by evaluation of the system about a normal operating point.

Thermal systems are distributed parameters and models involve partial differential equations. Models here will have lumped parameters so that ODE or Transfer Functions can be used.
**Liquid Level Systems**

Consider a tank with cross-sectional area, $A$, with liquid level height, $h$, and exit fluid resistance, $R$, density, $\rho$, Volume flow in, $q_i$, Volume flow out, $q_o$. The input is $q_i$ and output measured by $h$.

Then

$$\frac{d}{dt}(\rho Ah) = \rho q_i - \rho q_o$$

The mass flow out can be expressed in terms of $h$. Assume a linear model, the resistance of the exit orifice resistance of the valve $R$ is

$$\rho q_o = \frac{\Delta p}{R}$$
Liquid Level Systems

The pressure across the orifice (valve) is $\Delta p$. The mass flow rate out of the orifice is $\rho q_o$.

$$\Rightarrow \rho q_o = \frac{p_1 - p_2}{R} = \frac{(P_a + \rho gh) - P_a}{R} = \frac{\rho gh}{R}$$

*where* 1 – upstream, 2 – downstream

$p_1$ – hydrostatic, $p_2$ – atmospheric – $p_a$

*If the tank has constant cross section, $A$, and assume density, $\rho$, is constant, then*

$$\rho A \frac{dh}{dt} = \rho q_i - \frac{\rho gh}{R}$$
Liquid Level Systems

\[ \rho A \frac{dh}{dt} = \rho q_i - \frac{\rho gh}{R} \]

in differential form is

\[ A \dot{h} + \frac{g}{R} h = q_i \]

OR

\[ \frac{RA}{g} \dot{h} + h = \frac{R}{g} q_i \]

First order differential equation with time constant

\[ \tau = \frac{RA}{g} \]

Note: In this problem, we are assuming that we are operating about some set point and are linearizing the problem.

Note: The head versus flow rate is not a linear function.
Example - Two Tank System

The law of conservation of mass is applied to tank 1:

\[
\frac{d}{dt}(\rho A_1 h_1) = \rho q_i - \rho q_1
\]

\[
= \rho q_i - \frac{p_1 - p_2}{R_1}
= \rho q_i - \frac{(p_a + \rho g h_1) - (p_a + \rho g h_2)}{R_1}
\]

Thus,

\[
A_1 \dot{h}_1 + \frac{gh_1}{R_1} - \frac{gh_2}{R_1} = q_i \quad \text{(System Diff. Eq. 1)}
\]
Example - Two Tank System cont. 6-4

Conservation of mass is applied to tank 2:

\[
\frac{d}{dt}(\rho A_2 h_2) = \rho q_1 - \rho q_2
\]

\[
= \frac{p_1 - p_2}{R_1} - \frac{p_2 - p_3}{R_2}
\]

\[
= \frac{(p_a + \rho gh_1) - (p_a + \rho gh_2)}{R_1} - \frac{(p_a + \rho gh_2) - p_a}{R_2}
\]

Thus,

\[
A_2 \dot{h}_2 - \frac{gh_1}{R_1} + \left(\frac{1}{R_1} + \frac{1}{R_2}\right)gh_2 = 0
\]

(System Diff. Eq. 2)
# Mechanical Liquid-Level Analogy

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<tr>
<td>Viscous Friction</td>
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<td>Spring Constant</td>
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<td>Displacement</td>
<td>(x)</td>
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<tr>
<td>Velocity</td>
<td>(\dot{x})</td>
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Mechanical Liquid-Level Analogy

\[ q_1 \rightarrow c_1 \rightarrow h_1 \rightarrow q_1 \]

\[ h_1 \rightarrow c_1 \rightarrow q_1 \rightarrow h_2 \rightarrow q_2 \]

\[ F \rightarrow c_1 \rightarrow k_1 \rightarrow x_1 \]

\[ k_2 \rightarrow c_2 \rightarrow x_2 \]
Thermal Systems: Heat Transfer

Fundamentals

Heat transfer is a branch of the thermal sciences. It can be argued that there are only two basic and distinct modes in heat transfer: conduction and radiation (Rohsenow and Choi, 1961). Conduction is the transfer of heat by molecular motion. In both solids and fluids (gases and liquids), heat is conducted by elastic collision of molecules, and energy diffusion process. The word convection implies fluid motion. On the other hand, radiation, or more precisely thermal radiation, belongs to the general category of electromagnetic radiation. In thermal radiation, the energy is transmitted mainly in infrared waves (infrared region). The mechanisms of heat transfer anywhere in the fluid, even within solid bodies, are only conduction and radiation. The processes of conduction and radiation occur simultaneously. In many engineering problems, however, either conduction or radiation is dominant.

It is common practice, however, that the modes of heat transfer are conveniently put into three groups: conduction (diffusion through a substance), convection (fluid transport), and thermal radiation (electromagnetic radiation in the infrared region).
Thermal Systems: Heat Transfer

Conduction

In general, heat transfer, in an arbitrary x-direction, is governed by the Fourier law of conduction or Fourier's equation as

\[
\frac{q}{A_s} = -k \frac{\partial T}{\partial x}
\]

where the subscript \( s \) denotes surface, and the minus sign indicates that the heat flow is in the direction of decreasing temperature, i.e., negative \( \frac{\partial T}{\partial x} \) for positive \( \frac{q}{A_s} \). The equation was named after Fourier, although both Biot and Fourier had suggested it. In this equation,

- \( k \) = thermal conductivity of the solid or the fluid, Btu/(hr-ft-F)
- \( T \) = temperature, °F
- \( x \) = coordinate for an arbitrary x-direction, °F
- \( \frac{q}{A_s} \) = heat transfer rate per unit area, Btu/(hr-ft²)
Thermal Systems: Heat Transfer

Convection

Convection usually refers to heat transfer in fluids (gas and liquids). In this heat transfer for fluids at the solid-fluid interface is still governed by Fourier’s equation as

\[
\left( \frac{q}{A} \right)_s = -k_f \left( \frac{\partial T_f}{\partial x} \right)_s
\]

where the subscript \( f \) indicates fluid, and \( s \), again, stands for the surface of the solid-fluid interface. It is often difficult to evaluate the fluid temperature distribution, \( T_f \), or the partial derivative \( \frac{\partial T_f}{\partial x} \) at the surface. It is, however, more practical to determine the heat transfer for fluid by using Newton’s equation which was first suggested by Newton as

\[
\left( \frac{q}{A} \right)_s = h(T_s - T_f)
\]
Thermal Systems: Heat Transfer

Convection

In the equation

\[ \left( \frac{q}{A} \right)_s = h(T_s - T_f) \]

- \( h \) = surface coefficient of heat transfer, Btu/(hr-ft\(^2\)-F)
- \( T_s \) = surface temperature, °F
- \( T_f \) = fluid temperature, °F

\( \left( \frac{q}{A} \right)_s \) = rate of heat transfer per unit area, evaluated at the surface, Btu/(hr-ft\(^2\))

The surface coefficient of heat transfer \( h \), for many engineering problems, can be found in tabulated tables in literature.
Thermal Systems: Heat Transfer

Convection

A few words must be said about the fluid temperature. If the fluid is infinite, \( T_f \) is the fluid temperature at a distance far away from the surface: \( T_f = T_\infty \). If the fluid is flowing in a conduit such as a pipe, \( T_f \) is the mixed-mean temperature \( T_f = T_\infty \) which is defined as

\[
T_m = T_s - \frac{\int_0^R (T_s - T)v2\pi r dr}{\int_0^R v2\pi r dr}
\]

where \( v = v(r) \) and \( T = T(r) \) are the fluid velocity and temperature, respectively; both are functions of radius \( r \); \( R \) is the pipe radius.
Example – Air Heating System

Assume small deviations from steady state operation.
Assume loss to outside is negligible.

\[ \bar{\theta}_i = \text{steady state input air, } ^\circ C \]
\[ \bar{\theta}_0 = \text{steady state output air, } ^\circ C \]
\[ G = \text{mass flow rate of air through heating chamber, } \text{kg/s} \]
\[ M = \text{mass of air in chamber, } \text{kg} \]
\[ c = \text{specific heat of air, } \text{kcal/kg} - ^\circ C \]
\[ R = \text{thermal resistance, } ^\circ C - \text{s/kcal} \]
\[ C = \text{thermal capacitance of air in heating chamber, } M_c, \text{ kcal/ } ^\circ C \]
\[ \bar{H} = \text{steady state heat input, kcal/s} \]
Example - Air Heating System

Assume that the heat in is suddenly changed from \( H \) to \( H + h \) and the temperature in changes from \( \theta_i \) to \( \theta_i + \theta_i \) then the output air changes from \( \theta_0 \) to \( \theta_0 + \theta_0 \).

The equation to describe this behavior is

\[
C \frac{d\theta_o}{dt} = [h + GC(\theta_i - \theta_o)]dt
\]

which can be rearranged as

\[
C \frac{d\theta_o}{dt} = h + GC(\theta_i - \theta_o)
\]
Example – Air Heating System

Note that \( GC = \frac{1}{R} \) and substitute

\[
C \frac{d\theta_o}{dt} = h + \frac{1}{R} (\theta_i - \theta_o)
\]

\[
C \frac{d\theta_o}{dt} + \frac{1}{R} \theta_o = h + \frac{1}{R} \theta_i
\]

OR

\[
RC \frac{d\theta_o}{dt} + \theta_o = Rh + \theta_i
\]

First-order ODE system

- Time Constant \( \tau = RC \)
- Force (Input) \( = Rh \)
- Initial Condition \( = \theta_i \)
\[ \frac{q}{A_s} = -k \frac{\partial T}{\partial x} \]

**Convection**

\[ \left( \frac{q}{A} \right)_s = -k_f \left( \frac{\partial T_f}{\partial x} \right)_s \]