Problem #1 (15 points)

Solve the following IVP: \( x^2 y' = \frac{y^3}{2} \) with \( y(1) = 2 \).

The given d.e. is separable: \( x^2 \frac{dy}{dx} = \frac{y^3}{2} \Rightarrow \frac{2}{y^3} \, dy = \frac{1}{x^2} \, dx \Rightarrow \int 2y^{-3} \, dy = \int x^{-2} \, dx \Rightarrow -y^{-2} = -x^{-1} + c \).

\( y(1) = 2 \Rightarrow -2^{-2} = -1^{-1} + c \Rightarrow -1/4 = -1 + c \Rightarrow c = 3/4. \) Therefore, \(-y^{-2} = -x^{-1} + 3/4 = \frac{3x-4}{4x} \Rightarrow y^{-2} = \frac{4}{4-3x} \Rightarrow y^2 = \frac{4}{4-3x} \Rightarrow y = 2 \sqrt{\frac{x}{4-3x}} \)

Problem #2 (10 points)

Solve the following ODE: \( xy' - 2y = x^3e^{-2x} \).

This is a linear d.e. because \( y \) and \( y' \) appear only to the first power, multiplied by functions of \( x \) alone. Divide by \( x \) to write the d.e. in standard form: \( y' - \frac{2}{x} y = x^2e^{-2x} \). The integrating factor is \( \rho(x) = e^{\int -\frac{2}{x} \, dx} = e^{-2\ln(x)} = x^{-2} \). Multiplying both sides of the standard form of the d.e. by the integrating factor, we have \( x^{-2}y' - 2x^{-3}y = e^{-2x} \Rightarrow \frac{d}{dx} \left[ x^{-2}y \right] = e^{-2x} \Rightarrow x^{-2}y = \int e^{-2x} \, dx = -\frac{1}{2}e^{-2x} + c \Rightarrow y = -\frac{x^2}{2}e^{-2x} + cx^2 \).

Problem #3 (10 points)

Solve the following ODE: \( y' = \frac{x^2 + 3y^2}{2xy} \).

This is a homogeneous d.e. \( (y' \) equals a rational function, and each term in the numerator and denominator have the same degree.) We introduce the new variable \( v = y/x \). In the d.e. we replace \( y' \) by \( v + \frac{dv}{dx} \) and we replace \( y \) by \( xv \): \( y' = \frac{x^2 + 3y^2}{2xy} \Rightarrow v + \frac{dv}{dx} = \frac{x^2 + 3(xv)^2}{2xv} \Rightarrow v + \frac{2v}{x^2v} \, dv = v + \frac{2v}{2v} \, dv = 1 + 3v^2 \Rightarrow x \frac{dv}{dx} = 1 + 3v^2 - v = \frac{1 + v^2}{2v} \Rightarrow x \, dv = \frac{1 + v^2}{2x} \, dx \Rightarrow \int \frac{2v}{1 + v^2} \, dv = \int \frac{1}{x} \, dx \Rightarrow \ln \left( 1 + v^2 \right) = \ln(x) + c_1 \Rightarrow 1 + v^2 = e^{\ln(x) + c_1} = e^{c_1} \cdot x = cx \Rightarrow v^2 = cx - 1 \Rightarrow (y/x)^2 = cx - 1 \Rightarrow y^2 = x^2(cx - 1) \).

Problem #4 (15 points)

A rabbit farmer raises rabbits for most of the local pet stores in the area. Let \( t \) denote time in years, and let \( P(t) \) denote the number of rabbits on the farm at time \( t \). Under current breeding conditions at the farm, the birth rate per unit population is given by \( \beta = 2.22 - 0.001P \).

Assume that the farm is well managed so that the death rate is only 0.02 deaths per year per rabbit (i.e. 2 deaths occur per year for every 100 rabbits). Also assume that the rabbit farm supplies 210 rabbits per year to the local marketplace.

\[ \gamma = 0.02 \cdot \frac{100}{1000} = 0.002 \text{ deaths per year per rabbit}. \]
a. With the above description, formally develop and explain a mathematical model that describes the rabbit population versus time. Show that this model can be written, after some manipulation, as

\[
\frac{dP}{dt} = -0.001 (P - 100) (P - 2100)
\]

\[
\frac{dP}{dt} = \text{rate in} - \text{rate out} = \beta P - (\delta P + 210)
\]

\[
= (2.22 - 0.001P) P - (0.02P + 210)) = 2.22P - 0.001P^2 - 0.02P - 210
\]

\[
= -0.001P^2 + 2.2P - 210 = -0.001 (P^2 - 2200P + 210000) = -0.001 (P - 100) (P - 2100).
\]

b. Draw the phase line for the ODE given in Part a.

First we find the critical points: 

\[
-0.001 (P - 100) (P - 2100) = 0 \Rightarrow P = 100 \text{ or } P = 2100.
\]

The two critical points divide the phase line into 3 intervals: 

\[ P > 2100, \quad 100 < P < 2100, \quad \text{and} \quad P < 100. \]

and \[ P < 100. \]

\[
\left. \frac{dP}{dt} \right|_{P=2200} = -0.001 (2200 - 100) (2200 - 2100) < 0, \text{ so the direction arrow points down in the interval } P > 2100.
\]

\[
\left. \frac{dP}{dt} \right|_{P=1500} = -0.001 (1500 - 100) (1500 - 2100) > 0, \text{ so the direction arrow points up in the interval } 100 < P < 2100.
\]

\[
\left. \frac{dI}{dt} \right|_{P=50} = -0.001 (50 - 100) (50 - 2100) < 0, \text{ so the direction arrow points down in the interval } P < 100.
\]

c. Determine the stability of the critical points for this autonomous system.

From the phase line we can see that \[ 2100 \text{ is stable and } 100 \text{ is unstable.} \]

d. Carefully sketch the solution curves for \[ P(0) = 500, \ 1500, \text{ and } 2500. \]

Since 100 and 2100 are critical points, the horizontal lines \[ P = 100 \text{ and } P = 2100 \] are solution curves. From the phase line we see that \[ P(t) \to 2100 \text{ for both } P > 2100 \text{ and } 100 < P < 2100, \] so the solution curves for \[ P(0) = 500, \ 1500, \text{ and } 2500 \] all approach the horizontal line \[ P = 2100 \] as \[ t \] increases. See the figure above.

e. If \[ P(0) = 600 \text{ rabbits, what value does } P(t) \text{ approach as } t \text{ becomes large?} \]

From the phase line we can see that \[ P(t) \to 2100 \text{ as } t \text{ increases.} \]
Problem #6 (15 points)

Find the general solution to the following linear ODEs:

a. $y'' - 2y' + 10y = 0$ (2nd order system)

Characteristic equation: $r^2 - 2r + 10 = 0 \Rightarrow r = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(10)}}{2(1)} = \frac{2 \pm \sqrt{-36}}{2} = \frac{2 \pm 6i}{2}$

$= 1 \pm 3i \Rightarrow y = c_1e^x \cos(3x) + c_2e^x \sin(3x)$.

b. $y^{(4)} - y = 0$ (4th order system). Hint: $r^4 - 1 = (r^2)^2 - 1$.

Characteristic equation: $r^4 - 1 = 0 \Rightarrow (r^2 - 1)(r^2 + 1) = 0 \Rightarrow (r - 1)(r + 1)(r^2 + 1) = 0 \Rightarrow r = -1, 1, \pm i \Rightarrow y = c_1e^{-x} + c_2e^x + c_3 \cos(x) + c_4 \sin(x)$.

Problem #7 (10 points)

Find the solution to the following IVP:

$y''' - 4y'' - 5y' = 9 + 5x$ with $y(0) = 0$, $y'(0) = 0$, and $y''(0) = 4$.

1. Find $y_c$ by solving $y''' - 4y'' - 5y' = 0$. Characteristic equation: $r^3 - 4r^2 - 5r = 0 \Rightarrow r(r + 1)(r - 5) = 0 \Rightarrow r = 0, -1, 5 \Rightarrow y_c = c_1 + c_2e^{-x} + c_3e^{5x}$.

2. Find $y_p$. Since the right-hand side of the given d.e. $(9 + 5x)$ is a polynomial of degree 1, we guess a polynomial of degree 1: $y_p = Ax + B$. The constant $B$ in this guess duplicates the term $c_1$ in $y_c$, so we multiply our guess by $x$: $y_p = Ax^2 + Bx$. $y = Ax^2 + Bx \Rightarrow y' = 2Ax + B \Rightarrow y'' = 2A \Rightarrow y''' = 0$. $y''' - 4y'' - 5y' = 9 + 5x \Rightarrow 0 - 4(2A) - 5(2Ax + B) = 5 + 9x \Rightarrow -10Ax + (-8A - 5B) = 9 + 5x \Rightarrow -10A = 5$ and $-8A - 5B = 9 \Rightarrow A = -1/2$, $B = -1$. Therefore, $y_p = -x^2/2 - x$.

3. $y = y_c + y_p = c_1 + c_2e^{-x} + c_3e^{5x} - x^2/2 - x$.

4. $y = c_1 + c_2e^{-x} + c_3e^{5x} - x^2/2 - x \Rightarrow y' = -c_2e^{-x} + 5c_3e^{5x} - 1 \Rightarrow y'' = c_2e^{-x} + 25c_3e^{5x} - 1$.

$y(0) = 0 \Rightarrow 0 = c_1 + c_2e^0 + c_3e^0 - 0^2/2 - 0 \Rightarrow c_1 + c_2 + c_3 = 0$.

$y'(0) = 0 \Rightarrow 0 = c_2e^0 + 5c_3e^0 - 0 - 1 \Rightarrow -c_2 + 5c_3 = 1$.

$y''(0) = 4 \Rightarrow 4 = c_1e^0 + 25c_2e^0 - 1 \Rightarrow c_1 + 25c_3 = 5$.

Solving the last two equations for $c_2$ and $c_3$, we find $c_2 = 0$ and $c_3 = 1/5$. From the first equation we have $c_1 = -c_2 - c_3 = -1/5$. Therefore, $y = \frac{-1}{5} + \frac{1}{5}e^{5x} - \frac{1}{2}x^2 - x$.

Problem #8 (10 points)

a. Use the Laplace Transform to solve the following IVP:

$x'' + 4x' + 8x = 0$ with $x(0) = 0$ and $x'(0) = 2$.

$x'' + 4x' + 8x = 0 \Rightarrow \mathcal{L}\{x'' + 4x' + 8x\} = \mathcal{L}\{0\} \Rightarrow \mathcal{L}\{x''\} + 4\mathcal{L}\{x'\} + 8\mathcal{L}\{x\} = 0$,

$\Rightarrow s^2\mathcal{L}\{x\} - sx(0) - x'(0) + 4[s\mathcal{L}\{x\} - x(0)] + 8\mathcal{L}\{x\} = 0$,

$\Rightarrow s^2\mathcal{L}\{x\} - 0s - 2 + 4s\mathcal{L}\{x\} - 4(0) + 8\mathcal{L}\{x\} = 0 \Rightarrow$
\[
\left(s^2 + 4s + 8\right) \mathcal{L}\{x\} = 2 \Rightarrow \mathcal{L}\{x\} = \frac{2}{s^2 + 4s + 8} \Rightarrow x = \mathcal{L}^{-1}\left\{\frac{2}{s^2 + 4s + 8}\right\} = \mathcal{L}^{-1}\left\{\frac{2}{(s + 2)^2 + 4}\right\} \Rightarrow x = e^{-2t}\sin(2t)\] (from the Laplace transform table).

b. Find the Inverse Laplace Transform of the following function:

\[
F(s) = \frac{2s + 4}{s^2 + 4s - 5}
\]

Use partial fractions to simplify this expression. \(s^2 + 4s - 5 = (s - 1)(s + 5)\), so
\[
\frac{2s + 4}{(s - 1)(s + 5)} = \frac{A}{s - 1} + \frac{B}{s + 5}. \quad \text{Multiplying both sides of this equation by } (s - 1)(s + 5),
\]
we obtain
\[
2s + 4 = A(s + 5) + B(s - 1) = (A + B)s + (5A - B) \Rightarrow A + B = 2, \quad 5A - B = 4
\]
\[
\Rightarrow A = 1, \quad B = 1.
\]
Therefore,
\[
\mathcal{L}^{-1}\left\{\frac{2s + 4}{s^2 + 4s - 5}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s - 1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s + 5}\right\} = e^t + e^{-5t}
\]

**Problem #8 (15 points)**

The dynamics of a particular mechanical system are described by the IVP

\[
ymy'' + cy' + ky = f(t) \quad \text{with } y(0) = y_0 \quad \text{and } y'(0) = v_0
\]

where \(y(t)\) represents the deviation of the mass from equilibrium, \(f(t)\) is an applied force, and \(y_0\) and \(v_0\) represent the initial position and velocity of the mass, respectively. Within the context of this system, answer the following questions. Show your work and explain the logic used.

a. If the measured response of the unforced system with \(m = 5\) kg is

\[
y_c(t) = 2e^{-t}\cos\left(\sqrt{3}t - 1.0472\right)
\]

what are the numerical values for the damping coefficient \(c\) and spring constant \(k\) for this system?

What are the units of \(c\) and \(k\)? Assume that time is measured in seconds and that position is given in meters. (Hint: By looking at \(y_c\), can you tell what the roots of the characteristic equation must be?)

\(y_c\) is the solution of the d.e. \(my'' + cy' + ky = 0\). The characteristic equation for this d.e. is \(mr^2 + cr + k = 0\). Since \(m = 5\), the characteristic equation becomes \(5r^2 + cr + k = 0\). Since \(y_c\) contains a trig function, the roots of the characteristic equation must be complex: \(r = \alpha \pm \beta i\), where \(\alpha = -1\) (the coefficient of \(t\) in the exponential term) and \(\beta = \sqrt{3}\) (the coefficient of \(t\) in the cosine term). By the quadratic formula, we see that the roots of the characteristic equation are

\[
r = \frac{-c \pm \sqrt{c^2 - 4(5)(k)}}{2(5)} = \frac{-c \pm \sqrt{-20k - c^2}}{10} = \frac{-c \pm \sqrt{20k - c^2}i}{10}.
\]

Therefore,

\[
-1 = -\frac{c}{10} \quad \text{and} \quad \sqrt{3} = \frac{\sqrt{20k - c^2}}{10} \Rightarrow \left[c = 10 \text{ and } k = 20\right].
\]

Since all terms in the given d.e. have units of force (N), \(k\) must have units of N/m and \(c\) must have units of N/(m/s) = N s/m.

b. What initial conditions were required to give the measured response in Part a? In other words, what are the values of \(y_0\) and \(v_0\)?

\[
y_c(t) = 2e^{-t}\cos\left(\sqrt{3}t - 1.0472\right) \Rightarrow y_c(0) = 2e^0\cos\left(\sqrt{3}(0) - 1.0472\right) \approx 1.
\]

\[
y_c(t) = 2e^{-t}\cos\left(\sqrt{3}t - 1.0472\right) \Rightarrow y_c'(t) = -2e^{-t}\cos\left(\sqrt{3}t - 1.0472\right) - 2\sqrt{3}e^{-t}\sin\left(\sqrt{3}t - 1.0472\right) \Rightarrow y_c'(0) = -2e^0\cos\left(\sqrt{3}(0) - 1.0472\right) - 2\sqrt{3}e^0\sin\left(\sqrt{3}(0) - 1.0472\right) \approx 2. \text{ Therefore, } [y_0 = 1 \text{ and } v_0 = 2]}
\]
c. Estimate how long you would have to wait for the transient solution \( y_c \) to decay away to less than 1% of its original value.

1% of \( y_0 \) is 0.01. \( 2e^{-t} = 0.01 \Rightarrow e^{-t} = 0.005 \Rightarrow t = -\ln(0.005) \approx 5.3 \), so it takes approximately 5.3 seconds for the solution to decay away to less than 1% of its original value.

d. If the system is excited with a forcing function \( f(t) = 10\sin(4t) \), what is the frequency of the steady periodic solution? Explain. (Hint: Do you really need to find the steady periodic solution?)

The steady periodic solution is the particular solution \( y_p \). Since \( f(t) = 10\sin(4t) \), \( y_p \) will have the form \( y_p = A\cos(4t) + B\sin(4t) \). Thus, the period of the steady periodic solution is \( T = 2\pi/4 = \pi/2 \), so the frequency of the steady periodic solution is \( f = 1/T = 2/\pi \).