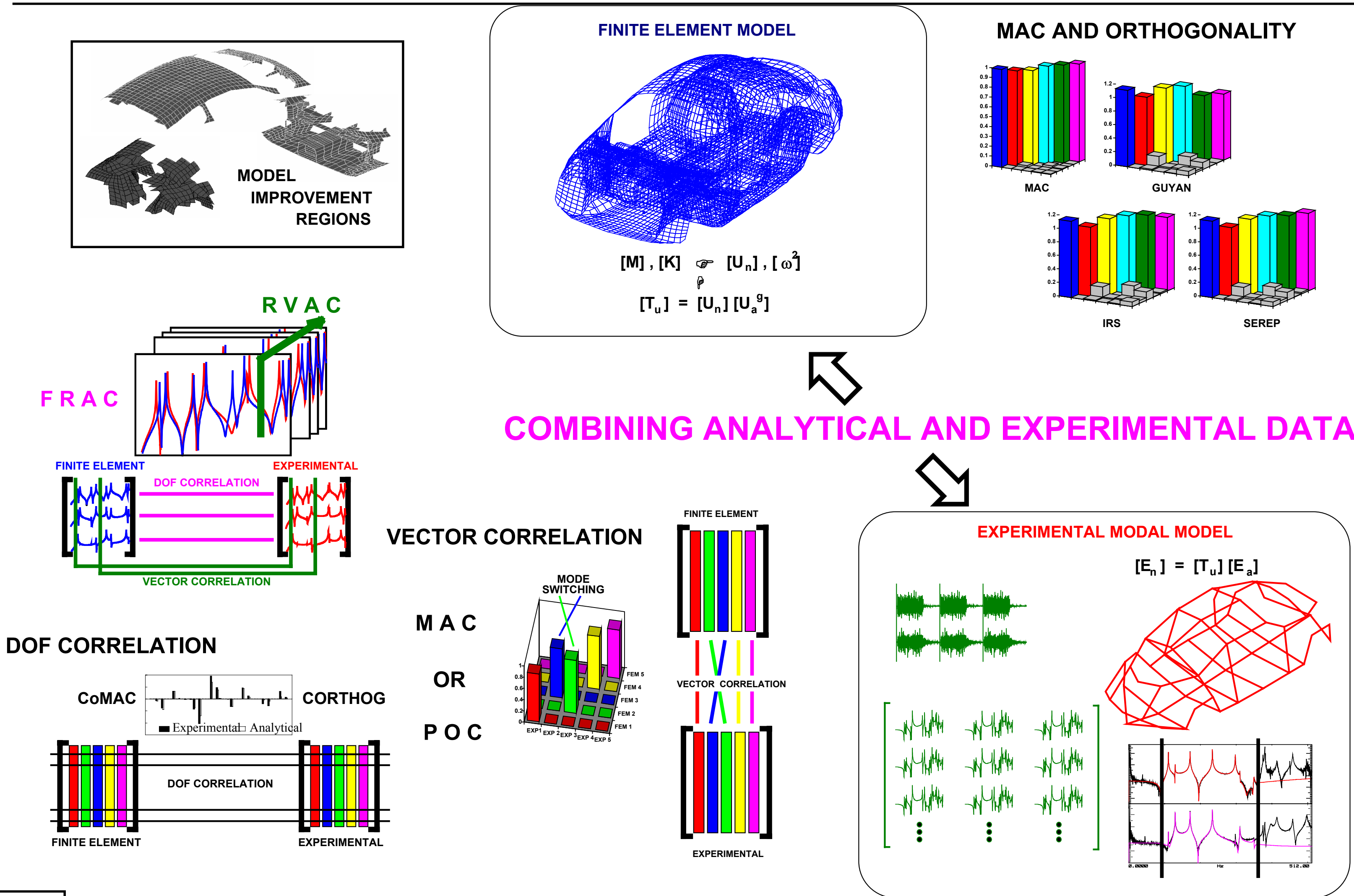
Test-Analysis Correlation-Updating Considerations

Peter Avitabile
Modal Analysis and Controls Laboratory
University of Massachusetts Lowell

The Overall Correlation and Updating Process



Test-Analysis Correlation-Updating Considerations

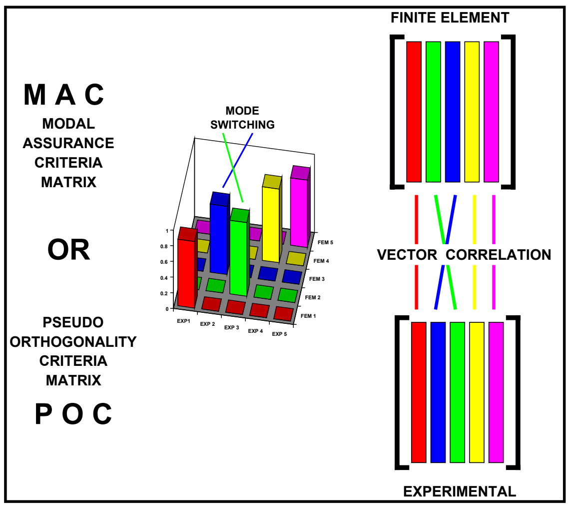
Objectives of this lecture:

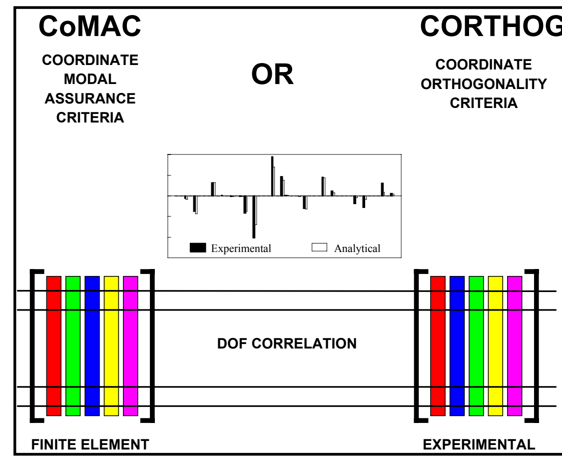
- ***Briefly describe the different correlation tools available***
- ***Conceptually, overview the correlation process***
- ***Briefly overview the model updating process***

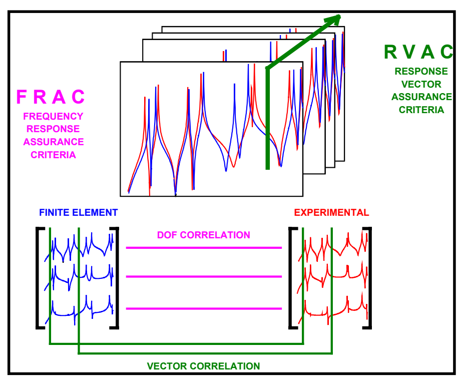
- ***A significant amount of effort is required to completely describe all the techniques and tools available***

Correlation Techniques

Correlation between analytical and experimental data is an important part of the structural dynamic characterization and updating of systems







Overview of Correlation Techniques

Vector correlation provides global indicator:

- **Modal Assurance Criteria**
- **Orthogonality Checks**

DOF correlation provides spatial indicator:

- **Coordinate Modal Assurance Criteria**
- **Coordinate Orthogonality Check**
- **Frequency Response Assurance Criteria**

Other tools:

- **MAC Contribution**
- **Force Unbalance**

Overview of Correlation Techniques

Two basic levels of correlation are considered:

- ***Modal vector correlation provides a global indicator of the level of correlation achieved***

- ***Degree of freedom (dof) correlation provides an indicator as to how the individual dofs contribute to the overall modal vector correlation***

Overview of Correlation Techniques

Vector Correlation Techniques:

Modal Assurance Criteria (MAC):

- Simple dot product
- independent of mass weighting

Orthogonality Checks (POC):

- Performed at 'n' space or 'a' space
- mass reduced for 'a' space calc
- shape expanded for 'n' space calc
- reduction/expansion has an effect

Overview of Correlation Techniques

DOF Correlation Techniques:

Coordinate Modal Assurance Criteria (CoMAC):

- Simple dot product
- correlation on dof basis for correlated mode pairs
- independent of mass weighting

Enhanced Coordinate Modal Assurance Criteria:

- Extension of CoMAC

Overview of Correlation Techniques

DOF Correlation Techniques:

Frequency Response Assurance Criteria (FRAC):

- simple dot product
- correlation of FEM and Test FRFs

Coordinate Orthogonality Check (CORTHOG)

- Identified correlation on a dof basis
- mass matrix used for weighting
- similar to CoMAC in concept except correlated mode pairs not required

Modal Assurance Criteria - MAC

Originally formulated for the test engineer to determine the degree of correlation between vectors from different tests, MAC between two vectors is defined as:

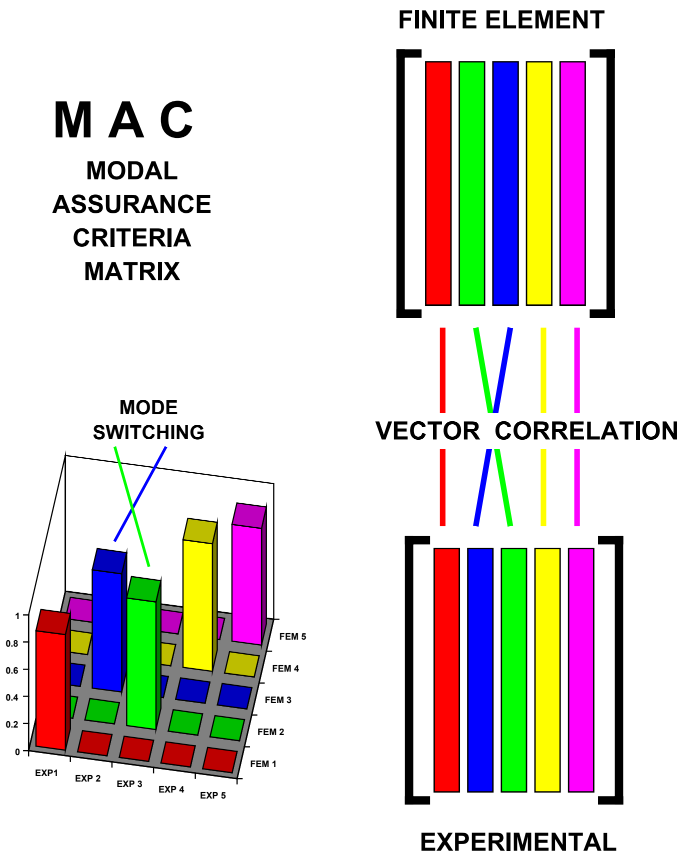
$$\mathrm{MAC}_{ij} = \frac{\left(\{V_i\}^T \{V_j\}\right)^2}{\left(\{V_i\}^T \{V_i\}\right)\left(\{V_j\}^T \{V_j\}\right)}$$

- ***values range between 0 and 1***
- ***approaching zero indicates no similarity***
- ***approaching one indicates high similarity***

Modal Assurance Criteria - MAC

MAC was extended to allow an assessment between analytical and experimental modal vectors:

$$MAC_{ij} = \frac{\left[\{u_i\}^T \{e_j\}\right]^2}{\left[\{u_i\}^T \{u_i\}\right]\left[\{e_j\}^T \{e_j\}\right]}$$



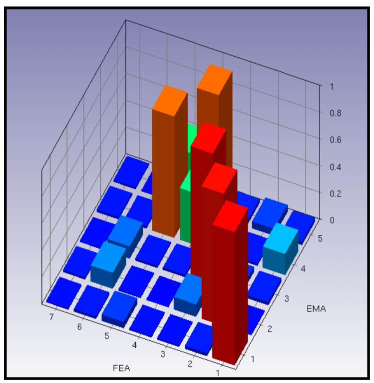
- ***low values - not similar***
- ***high values - very similar***

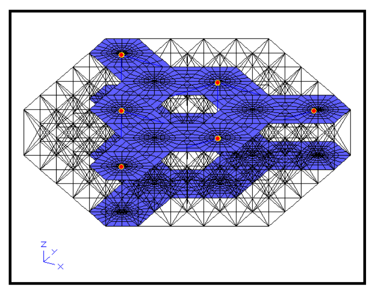
Orthogonality Check

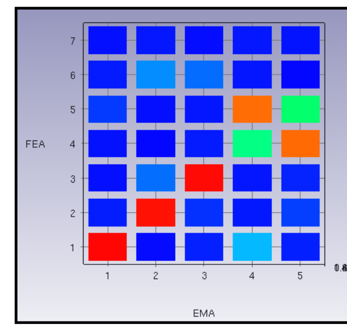
For modal vectors scaled to unit modal mass, the vectors must satisfy the orthogonality condition:

$$[U]^T [M][U] = [I]$$

$$[U]^T [K][U] = [\Omega^2]$$



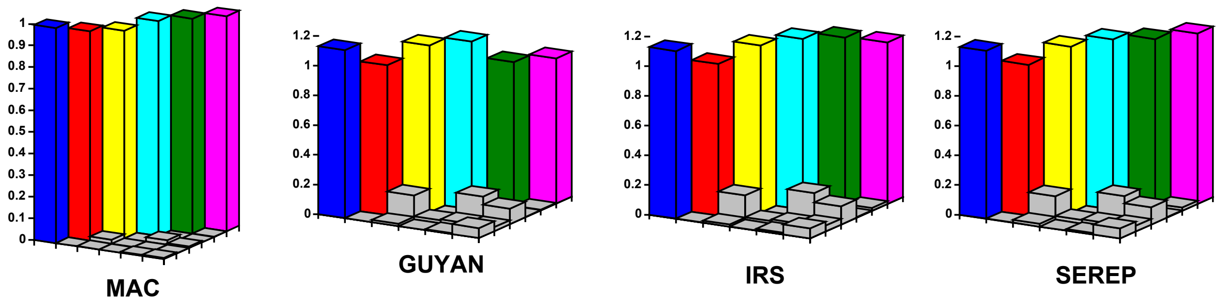




Pseudo Orthogonality Check - POC

The Pseudo Orthogonality Check relating the correlation between the analytical and experimental modal vectors with the analytical mass matrix is

$$POC = [E]^T\,[M][U] \stackrel{?}{=} [I]$$



Typically, most people feel the smaller the POC off-diagonal terms the better correlation that exists. However, these terms may be small and vectors may still be relatively uncorrelated

Pseudo Orthogonality Check - POC

The Pseudo Orthogonality Check is an assessment as to how close the experimental vectors are aligned with the analytical vectors

$$[E]^T [M][U] \stackrel{?}{=} [I] \qquad [E]^T [K][U] \stackrel{?}{=} [\Omega^2]$$

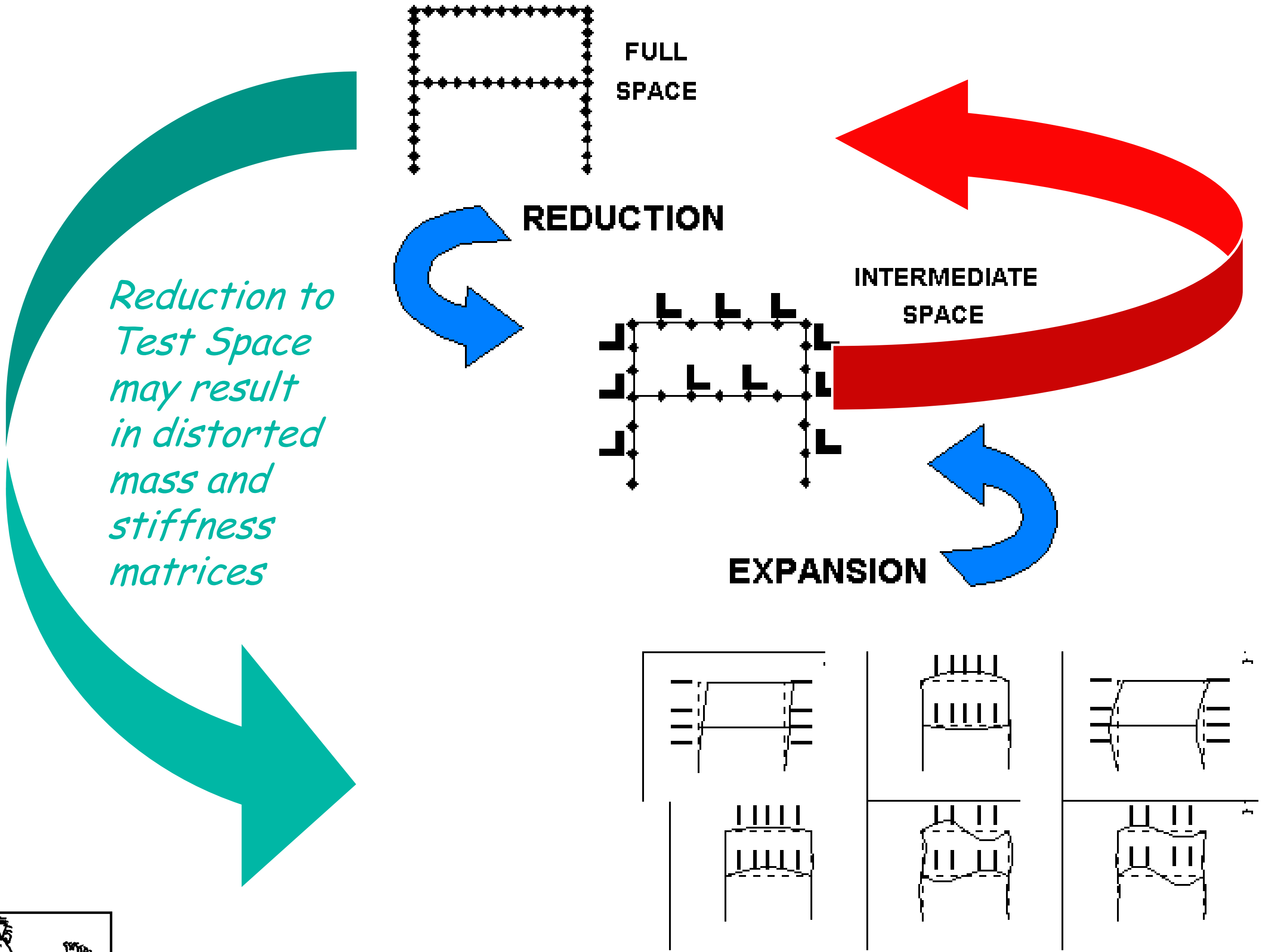
These equations can be evaluated at:

- ***FEM Space - requires expansion***
- ***Reduced Space - requires reduction***
- ***Intermediate space - requires both***

<u>Substantial numerical advantages using SEREP!</u>

Pseudo Orthogonality Check - POC

Reduction to Test Space may result in distorted mass and stiffness matrices



Expansion to Full Space may smear and distort mode shapes

Cross Orthogonality Check

The Cross Orthogonality Check is also used for correlation purposes

$$[E]^T \left[M\right][E] \overset{?}{=} [I] \qquad [E]^T \left[K\right][E] \overset{?}{=} [\Omega^2]$$

These equations can be evaluated at:

- ***FEM Space - requires expansion***
- ***Reduced Space - requires reduction***

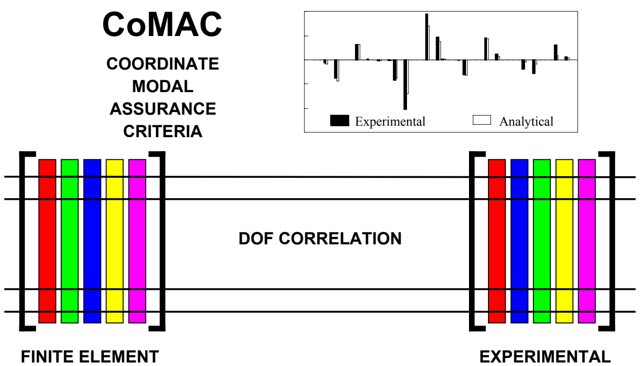
Similar to POC (off-diagonal terms are squared)

Coordinate Modal Assurance Criteria - CoMAC

The CoMAC gives an indication of the contribution of each dof to the MAC for a given mode pair

$$\mathrm{CoMAC}(k) = \frac{\left[\sum_{c=1}^{m} \left| u_k^{(c)} \cdot e_k^{(c)} \right| \right]^2}{\sum_{c=1}^{m} \left(u_k^{(c)} \right)^2 \cdot \sum_{c=1}^{m} \left(e_k^{(c)} \right)^2}$$

Low values of CoMAC indicate little correlation whereas high values of CoMAC indicate very high correlation

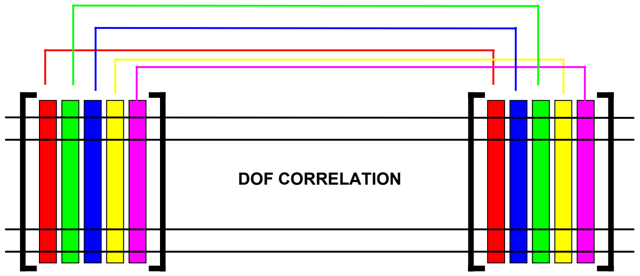


Modulus Difference

The Modulus Difference was developed to supplement the results from CoMAC

$$\text{ModulusDifference}(k) = \left| u_k^{(c)} \right| - \left| e_k^{(c)} \right|$$

Assists in identifying discrepancies between analytical and experimental vectors



Enhanced CoMAC - ECoMAC

The CoMAC gives an indication of the contribution of each dof to the MAC for a given mode pair

$$\mathrm{ECoMAC}(k) = \frac{\left[\sum_{c=1}^{m} \left| u_k^{(c)} - e_k^{(c)} \right| \right]}{2m}$$

Low values of ECoMAC indicate high correlation whereas high values of CoMAC indicate very low correlation

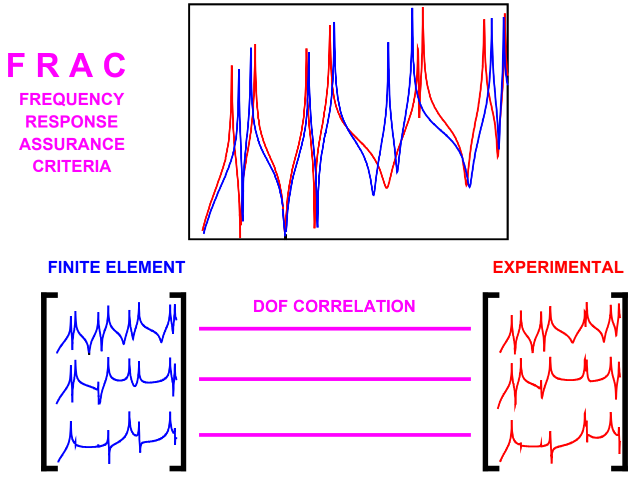
Very sensitive to phasing of vectors - which makes it more sensitive

Frequency Response Assurance Criteria - FRAC

The FRAC is used to identify similarity between a measured and analytical FRF - formed like MAC

$$\mathrm{FRAC}(j)=\frac{\left|\left\{H(\omega_i)_j^a\right\}\cdot\left\{H(\omega_i)_j^x\right\}^*\right|^2}{\left(\left\{H(\omega_i)_j^a\right\}\cdot\left\{H(\omega_i)_j^a\right\}^*\right)\cdot\left(\left\{H(\omega_i)_j^x\right\}\cdot\left\{H(\omega_i)_j^x\right\}^*\right)}$$

Low values of FRAC indicate little correlation whereas high values of FRAC indicate very high correlation

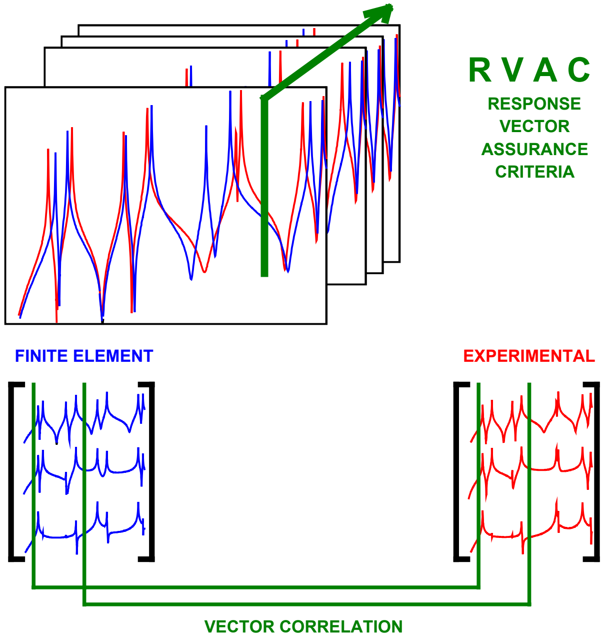


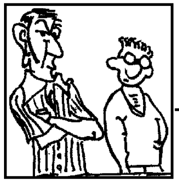
Response Vector Assurance Criteria - RVAC

The RVAC is used to identify the degree of similarity that exists at a particular frequency

$$\mathrm{RVAC}(\omega) = \mathrm{MAC}\left(\{E_{test}(\omega)\}, \{U_{fem}(\omega\beta)\}\right)$$

Low values of RVAC indicate little correlation whereas high values of RVAC indicate very high correlation





Coordinate Orthogonality Check - CORTHOG

The Coordinate Orthogonality Check helps to identify the contribution of individual dofs to each of the off-diagonal terms of the POC matrix

Identifies which dof are most discrepant between the analytical and experimental vectors on a mass weighted basis

POC

$$POC_{ij}^{k} = \sum_{p} e_{ki} m_{kp} u_{pj}$$

Orthogonality

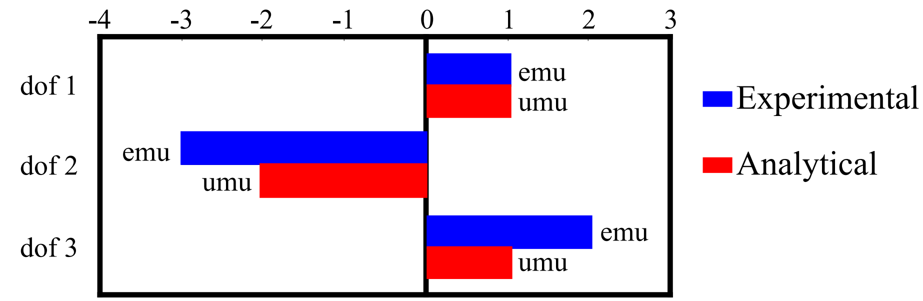
$$ORTHOG_{ij}^{k} = \sum_{p} u_{ki} m_{kp} u_{pj}$$

Coordinate Orthogonality Check - CORTHOG

The Coordinate Orthogonality Check is simply the comparison of what should have been obtained analytically for each dof in an orthogonality check to what was actually obtained for each dof in a pseudo-orthogonality check from test

$$SD = CORTHOG_{ij}^{k} = \sum_{p} e_{ki} m_{kp} u_{pj} - u_{ki} m_{kp} u_{pj}$$

Variety of different formulations with different scaling approaches



MAC Contribution

The MAC Contribution is a relatively simple and straightforward technique to determine the degree of contribution of each dof to the MAC value achieved

- ***pick a mode pair of interest***
- ***select a target MAC value***
- ***delete dof until target MAC value achieved***

Force Unbalance

The Force Balance is a simple calculation to determine the inequality that exists in the equation of motion

$$\left[[K]-\lambda[M]\right]\{x\}\overset{?}{=}\{0\}$$

- ***uses the FEM mass and stiffness matrices***
- ***uses experimental frequencies and mode shapes***
- ***compute the inequality that exists***

Model Updating Topics

Model Updating techniques can be broken down into two categories:

- ***Direct Techniques***
- ***Indirect Techniques (Sensitivity based)***

  Modal Based Techniques
  Response Based Techniques

Some basic theory of analytical model improvement and localization of model change are described

Model Improvement Terminology

Nomenclature

$[M_S]$ **finite element mass matrix (seed matrix)**

$[K_S]$ **finite element stiffness matrix (seed matrix)**

$[M_I]$ **updated mass matrix (target matrix)**

$[K_I]$ **updated stiffness matrix (target matrix)**

$[\Delta M]$ **differential mass matrix**

$[\Delta K]$ **differential stiffness matrix**

$[E]$ **target mode shapes**

$$[E]^g = \left[[E]^T[M][E]\right]^{-1}[E]^T[M] = \left[\overline{M}\right]^{-1}[E]^T[M]$$ **generalized inverse**

Analytical Model Improvement - AMI

Analytical Model Improvement – Direct Inversion

Using a direct inversion of the modal matrices to obtain physical space matrices can be performed. These equations can be written for the set of tested dof. The size of the physical matrices is controlled by the number of measurement dof.

The modal mass and modal stiffness in modal space can be written as

$$[E]^T[M][E] = [\overline{M}] = [I]$$

$$[E]^T[K][E] = [\overline{K}] = [\Omega^2]$$

Using a generalized inverse, the physical mass and stiffness can be written as

$$[E]^{Tg}[I][E]^g = [M]$$

$$[E]^{Tg}[\Omega^2][E]^g = [K]$$

where

$$[E]^g = [[E]^T[E]]^{-1}[E]^T$$

It is important to note that this mass and stiffness obtained from this approach are not expected to have any physical meaning whatsoever.

Analytical Model Improvement - AMI

Analytical Model Improvement – Mass Alone

Using the original finite element matrices as seeding matrices for the inverse process and for the development of a weighted generalize inverse, the following equations represent the updated system characteristics. These equations represent the updating of the mass independent of the stiffness matrix and equations of motion for the system.

The discrepancy of modal mass in modal space can be written in terms of the difference of the modal mass of the improved mass and the original modal mass as

$$\left[\Delta\overline{M}\right] = [E]^T[M_I][E] - [E]^T[M_S][E]$$

This can be projected from modal space back to physical space through the use of a mass weighted generalized inverse as

$$[\Delta M] = \left[\left[\overline{M}_S\right]^{-1}[E]^T[M_S]^T\right]^T\left[\Delta\overline{M}\right]\left[\left[\overline{M}_S\right]^{-1}[E]^T[M_S]\right]$$

where

$$[\Delta M] = [V]^T\left[\Delta\overline{M}\right][V] \quad \text{and} \quad [V] = \left[\left[\overline{M}_S\right]^{-1}[E]^T[M_S]\right]$$

The improved mass can then be written as

$$[M_I] = [M_S] + [V]^T\left[I - \overline{M}_S\right][V]$$

It is important to note that this mass is not necessarily related to the stiffness of the system since no equations of motion have been invoked.

Analytical Model Improvement - AMI

Analytical Model Improvement – Stiffness Alone

Using the original finite element matrices as seeding matrices for the inverse process and for the development of a weighted generalize inverse, the following equations represent the updated system characteristics. These equations represent the updating of the stiffness independent of the mass matrix and equations of motion for the system.

The discrepancy of stiffness can be projected from modal space back to physical space through the use of a mass weighted generalized inverse as

$$[\Delta K] = \left[[\overline{M}_S]^{-1}[E]^T[M_S]^T\right]^T [\Delta\overline{K}] \left[[\overline{M}_S]^{-1}[E]^T[M_S]\right]$$

where

$$[\Delta K] = [V]^T [\Delta\overline{K}] [V]$$

The improved stiffness can then be written as

$$[K_I] = [K_S] + [V]^T \left[`\omega^2 - \overline{K}_S \right] [V]$$

It is important to note that this stiffness is not necessarily related to the mass of the system since no equations of motion have been invoked.

Analytical Model Improvement - AMI

Analytical Model Improvement – Mass and Stiffness Together

Using the original finite element matrices as seeding matrices for the inverse process and for the development of a weighted generalize inverse, the following equations represent the updated system characteristics. These equations represent the updating of the system matrices which include the relationship of the equations of motion as an additional constraint in the inverse process. In order to solve these equations, a weighted generalized inverse invoking the common solution is required.

The force balance associated with the eigenvalue problem is given as

$$[K][E] = [M_I][E][{}^{\backslash}\omega^2{}_{\backslash}]$$

and is projected to modal space as

$$[E^T][K_I][E] = [E]^T[M_I][E][{}^{\backslash}\omega^2{}_{\backslash}]$$

Using the weighted generalized inverse for the common solution yields

$$[K_I] = [K_S] + [V]^T[{}^{\backslash}\omega^2 + \overline{K}_S{}_{\backslash}][V] - [[K_S][E][E]^T[M_I]] - [[K_S][E][E]^T[M_I]]^T$$

or

$$[K_I] = [K_S] + [V]^T[{}^{\backslash}\omega^2 + \overline{K}_S{}_{\backslash}][V] - [[K_S][E][V]] - [[K_S][E][V]]^T$$

It is extremely important to note that the equations of motion have been invoked as part of this updating process. The implication of this is that the eigensolution of the updated system matrices will contain the target modal vectors exactly as included in the process. In addition, the higher frequency original analytical modal vectors (above those of the target modes) will also be preserved exactly as included in the process.

Analytical Model Improvement - SSO

<u>Analytical Model Improvement –Stiffness Skyline Optimization (SSO)</u>

Using the original finite element matrices as seeding matrices for the inverse process and for the development of a weighted generalize inverse, the following equations can be developed which represent the updated system characteristics.

$$[K_I][E][E]^T[M_I] = [M_I][E]\left[{}^{\backslash}\omega^2{}_{\backslash}\right][E]^T[M_I]$$

The terms of the matrix equations can be re-packaged into vector form as

$$[K_I] \Rightarrow \{Y_K\}$$

$$[E][E]^T[M_I] \Rightarrow [B]$$

$$[M_I][E]\left[{}^{\backslash}\omega^2{}_{\backslash}\right][E]^T[M_I] \Rightarrow \{D\}$$

Due to symmetry, this equation can be written as

$$[B]\{Y_K\} = \{D\}$$

These can be partitioned into DOF that can and cannot change based on the skyline of the stiffness matrix as

$$\left[[B_p] \;\; \vdots \;\; [B_q]\right] \begin{Bmatrix} \{Y_p\} \\ \cdots \\ \{Y_q\} \end{Bmatrix} = \begin{Bmatrix} \{D_p\} \\ \cdots \\ \{D_q\} \end{Bmatrix}$$

The least squares solution of this equation yields the best improved matrix terms based on the constraint of the skyline imposed. These terms can then be transferred back to matrix form.

Analytical Model Improvement - MSSO

Analytical Model Improvement – Mass & Stiffness Skyline Optimization (MSSO)

Using the original finite element matrices as seeding matrices for the inverse process and for the development of a weighted generalize inverse, the following equations can be developed which represent the updated system characteristics.

$$[E][E]^T[K_I]-[E]\left[{}^{\backprime}\omega^2\right][E]^T[M_I]=0$$

The terms of the matrix equations above can be re-packaged into vector form as

$$[E][E]^T \Rightarrow [P] \qquad [K_I] \Rightarrow \{Y_k\}$$

$$[E]\left[{}^{\backprime}\omega^2\right][E]^T \Rightarrow [Q] \qquad [M_I] \Rightarrow \{Y_m\}$$

$$[E][M_I][E]^T=[R]\{Y_m\} \qquad [I] \Rightarrow \{V\}$$

Then the following can be written

$$\begin{bmatrix} [P] & [-Q] \\ [0] & [R] \end{bmatrix} \begin{Bmatrix} \{Y_k\} \\ \cdots \\ \{Y_m\} \end{Bmatrix} = \begin{Bmatrix} \{0\} \\ \cdots \\ \{V\} \end{Bmatrix}$$

The least squares solution of this equation yields the best improved matrix terms based of the constraint of the skyline imposed. These terms can then be transferred back to matrix form.

Model Updating - Sensitivity Approaches

Analytical Model Improvement – Sensitivity Approach

Using the basic equation of motion, the change in frequency can be given as

$$\left[[K]-\omega_r^2[M]\right]\{u_r\}=\{0\}$$

$$\frac{\partial\omega_r}{\partial p}=\frac{1}{2m_r\omega_r}\{u_r\}^T\frac{\partial[K]}{\partial p}\{u_r\}-\frac{\omega_r}{2m_r}\{u_r\}^T\frac{\partial[M]}{\partial p}\{u_r\}$$

The change in [M] and [K] due to the selected parameters that make up the elements
Parameters can be A, E, I, ρ, t, etc
or one global parameter for an individual element or group of elements
Design or proportional changes

Then

$$\omega_r^e-\omega_r^a=\frac{1}{2m_r\omega_r}\{u_r\}^T\frac{\partial[K]}{\partial p}\{u_r\}-\frac{\omega_r}{2m_r}\{u_r\}^T\frac{\partial[M]}{\partial p}\{u_r\}$$

$$\{q_t\}-\{q_c\}=[S]\{dp\}$$

$$\begin{Bmatrix}\text{target}\\\text{results}\end{Bmatrix}-\begin{Bmatrix}\text{current}\\\text{state}\end{Bmatrix}=\begin{bmatrix}\text{sensitivity}\\\text{matrix}\end{bmatrix}\begin{Bmatrix}\text{parameter}\\\text{changes}\end{Bmatrix}$$

Comments on Direct Techniques

Direct Techniques

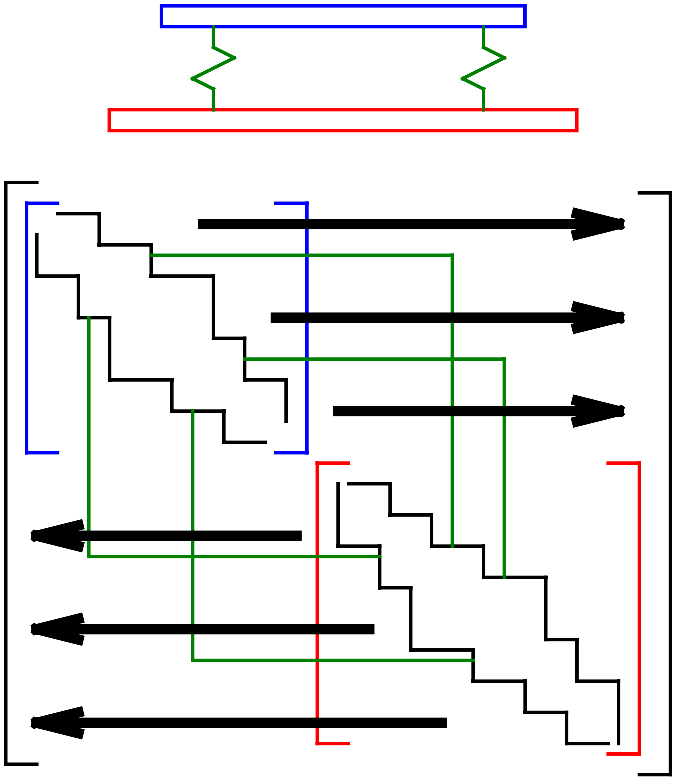
- *Usually a one step process that does not require iteration to obtain a solution*
- *Usually based on equation of motion and orthogonality conditions*

- *Exact results obtained (in the sense that the target modes are reproduced*
- *Generally updated matrices are difficult to interpret and smearing of results occurs*

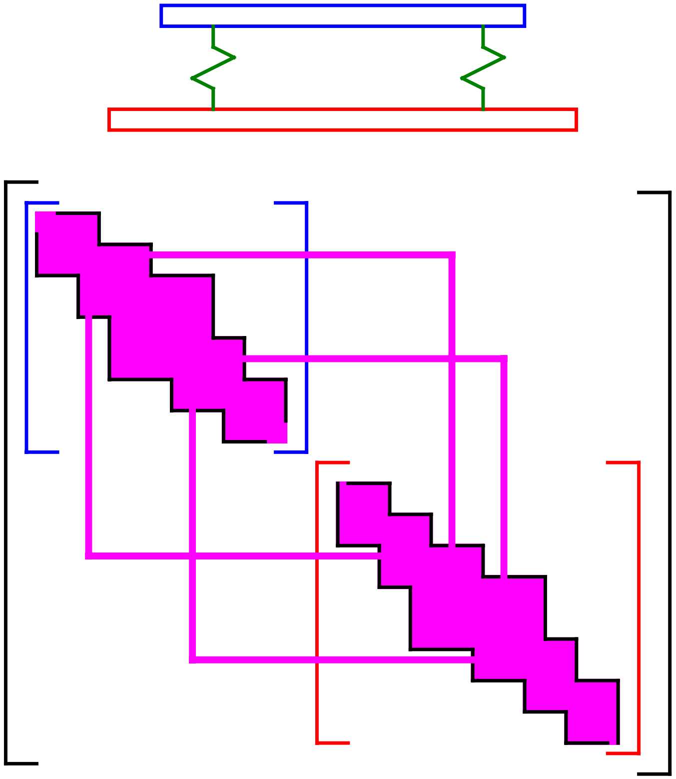
- *Skyline approaches attempt to retain the original topology of the system assembly*
- *Reduction and expansion have a dramatic effect on results*

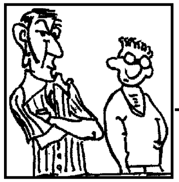
Comments on Direct Techniques

Matrix smearing



Skyline containment





Model Updating - Sensitivity Approaches

Differences that are typically minimized:

- *Frequency differences*
- *Mode shape differences*
- *Frequency response differences*

Parameters that may be updated:

- *mass/stiffness of individual elements*
- *mass/stiffness of groups of elements*
- *parameters associated with individual elements*
- *parameters associated with groups of elements*

Comments on Indirect Techniques

Indirect Techniques - Sensitivity approach

Modal Based Techniques

Frequency differences

Shape differences

Response differences

Comments on Indirect Techniques

Indirect Techniques -Sensitivity -Modal Approach

Frequency differences

- Likely to be the most accurate parameter measured
- No spatial information needed
- Relatively simple calculations
- No reduction/expansion problems

Comments on Indirect Techniques

Indirect Techniques -Sensitivity -Modal Approach

Shape differences

- Less accurate on a dof basis
- Spatial information included
- Mode pairing necessary
- Calculations more complicated
- Reduction/expansion is a problem

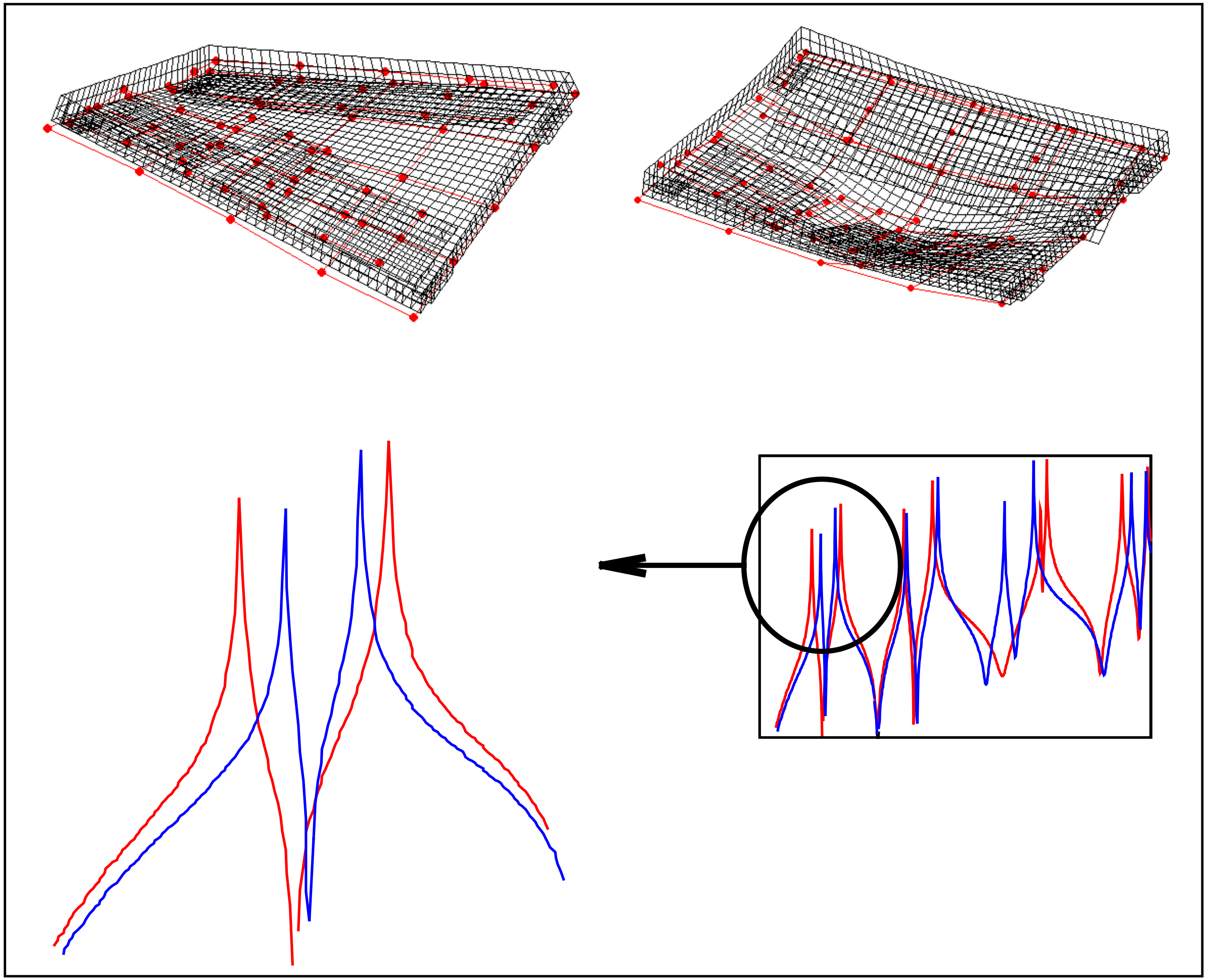
Comments on Indirect Techniques

Indirect Techniques - Sensitivity approach

Response Based Techniques

- Contains complete information in frequency range
- No need to estimate modal parameters
- FRFs are more accurate than modal parameters
- Response may be item of interest
- Damping may be difficult to determine
- Selection of certain spectral lines may cause numerical difficulties
- Using only a few FRFs may distort the results
- Difficult to identify parameters for change
- Measured FRFs must be acquired with high accuracy

General Comments



General Comments

- ***Use of all the correlation tools necessary to interpret the data available***
- ***Both modal and response based techniques should be used together for the updating***
- ***One technique alone may not be sufficient to adequately update the model***

- ***Once updated, the model should be perturbed both analytically and experimentally and the correlation process repeated to assure that meaningful parameters have been obtained from the updating process***

General Comments

- ***Model Updating requires extreme care in order to obtain reliable results***
- ***A firm understanding of the modeling techniques employed are necessary in order to adequately adjust the finite element model***
- ***A thorough understanding of the experimental data used for the updating process is critical***
- ***A clear definition of what is meant by an "improved" model is necessary***
- ***The analyst has a tremendous responsibility in identifying which areas of the model are to be updated and which sets of modes are the best modes to use in the updating process***