Impedance Modeling
&
Frequency Based Substructuring

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Impedance Based Modeling Considerations

Objectives of this lecture:

- **Describe the impedance modeling approach**
- **Extend this technique to system model development**
- **A significant amount of effort is required to completely describe all aspects of these techniques**
Impedance Modeling Techniques

The impedance modeling approach has been around for many years.

Its application becomes more popular as major breakthroughs occur in either

- measurement capability (16-24 bit ADC)
- or
- computational speed availability (PC systems)
Impedance Modeling Techniques

A frequency response function can be easily measured with high resolution on today’s digital data acquisition systems.

or computed easily from an analytical model

\[ H_{ij}(j\omega) = \sum_{k=1}^{m} \frac{q_k u_{ik} u_{jk}}{(j\omega - p_k)} + \frac{q_k u_{ik}^* u_{jk}^*}{(j\omega - p_k^*)} \]
Impedance Modeling Techniques

Consider a cantilever beam. It is desired to estimate the FRF between point c and b when the tip of the beam is pinned to ground.

In particular, the FRF $h_{cb}$ when $x_a = 0$
Impedance Modeling Techniques

The response at "a" is related to the force at "a" and "b" through

\[ x_a = h_{ab}f_b + h_{aa}f_a \]

where \( x_a \) is the vertical translation at the tip of the beam.

With the constraint \( x_a = 0 \), the force at point "a" becomes

\[ f_a = -h_{aa}^{-1}h_{ab}f_b \]
Impedance Modeling Techniques

The response at "c" due to an excitation at "a" and "b" is

\[ x_c = h_{ca} f_a + h_{cb} f_b \]

In order to include the effects of the constraint at "a", the force at point "a" with the constraint \( x_a = 0 \), changes this equation to

\[ \tilde{h}_{cb} = \frac{x_c}{f_b} = h_{cb} - h_{ca} h_{aa}^{-1} h_{ab} \]

which are obtained from the unconstrained system.
Summary of Impedance Modeling

Frequency Response Functions can also be used to investigate structural modifications. The FRF can be written as

$$H_{ij}(j\omega) = \sum_{k=1}^{m} \frac{q_k u_{ik} u_{jk}^*}{(j\omega - p_k)} + \frac{q_k u_{ik}^* u_{jk}}{(j\omega - p_k^*)}$$

Using force balance and compatibility equations, the effects of a modification can be written in terms of the unmodified system as

$$x_a = H_{ab} F_b + H_{aa} F_a$$
$$F_a = -H_{aa}^{-1} H_{ab} F_b$$
$$x_c = H_{ca} F_a + H_{cb} F_b$$

$$\tilde{H}_{cb} = \frac{x_c}{F_b} = H_{cb} - H_{ca} H_{aa}^{-1} H_{ab}$$
System Modeling Techniques

Consider combining two systems together

![Diagram of system modeling techniques]

COMPONENT (A)
- a-DOFs
- c-DOFs

COMPONENT (B)
- c-DOFs
- b-DOFs

SYSTEM (S)
System Modeling Techniques

The equation of motion for each component is

$$\{X\}_n = [H]_{nn} \{F\}_n$$

where

$$n = a + c$$ for component (A)
$$n = b + c$$ for component (B).

Note that the number of "c" coordinates are the same on component (A) and (B).
Component A can be partitioned as

\[
\begin{pmatrix}
X^A_A \\
X^A_c
\end{pmatrix}_n =
\begin{bmatrix}
H^A_{aa} & H^A_{ac} \\
H^A_{ca} & H^A_{cc}
\end{bmatrix}_{nn}
\begin{pmatrix}
F^A_A \\
F^A_c
\end{pmatrix}_n
\] (4-8)

Component B can be partitioned as

\[
\begin{pmatrix}
X^B_b \\
X^B_c
\end{pmatrix}_n =
\begin{bmatrix}
H^B_{bc} & H^B_{bb} \\
H^B_{cc} & H^B_{cb}
\end{bmatrix}_{nn}
\begin{pmatrix}
F^B_b \\
F^B_c
\end{pmatrix}_n
\] (4-9)
System Modeling Techniques

When rigidly connecting Component A to Component B, compatibility implies that

\[
\begin{bmatrix} \dot{X}_A \\ \dot{X}_B \end{bmatrix}_c = \begin{bmatrix} \dot{X}_S \end{bmatrix}_c
\]  \hspace{1cm} (4-10)

and equilibrium at the "c" DOFs requires that

\[
\begin{bmatrix} F_A \\ F_B \end{bmatrix}_c + \begin{bmatrix} F_S \end{bmatrix}_c
\]  \hspace{1cm} (4-11)

where "S" superscript is used to represent system comprised of Component A rigidly coupled to Component B at the connection DOFs "c"
System Modeling Techniques

The FRFs of the uncoupled system can be defined as

\[
\begin{bmatrix}
\{X^S\}_a \\
\{X^S\}_c \\
\{X^S\}_b
\end{bmatrix}_n =
\begin{bmatrix}
H^S_{aa} & H^S_{ac} & H^S_{ab} \\
H^S_{ca} & H^S_{cc} & H^S_{cb} \\
H^S_{ba} & H^S_{bc} & H^S_{bb}
\end{bmatrix}_{nn}
\begin{bmatrix}
\{F^S\}_a \\
\{F^S\}_c \\
\{F^S\}_b
\end{bmatrix}_n
\]

From the partitioned equations for Component A and Component B, the connection DOF are

\[
\begin{align*}
\{X^A\}_c &= \left[ H^A_{ca} \right] \{F^A\}_a + \left[ H^A_{cc} \right] \{F^A\}_c \\
\{X^B\}_c &= \left[ H^B_{cc} \right] \{F^B\}_c + \left[ H^B_{cb} \right] \{F^B\}_b
\end{align*}
\]
System Modeling Techniques

These two equations can be equated and used to solve for the connection force as

\[
\begin{align*}
\{ \tilde{F}^A \}_c &= \begin{bmatrix} H^A \end{bmatrix}_{cc} + \begin{bmatrix} H^B \end{bmatrix}_{cc}^{-1} \begin{bmatrix} H^B \end{bmatrix}_{cb} \{ F^B \}_b - \begin{bmatrix} H^A \end{bmatrix}_{ca} \{ F^A \}_a + \begin{bmatrix} H^B \end{bmatrix}_{cc} \{ F^S \}_c \\
\end{align*}
\]

From these equations derived above, the coupled system FRFs can be determined in terms of the uncoupled FRFs of the individual components. Equations (4-8), (4-9), (4-12) and (4-15) are used in the development of the coupled system.
System Modeling Techniques

As an example, \( [H^S]_{aa} \)

will be derived.

The first equation of (4-12) of the coupled system is

\[
\{X^S\}_a = [H^S]_{aa} \{F^S\}_a + [H^S]_{ac} \{F^S\}_c + [H^S]_{ab} \{F^S\}_b
\]  

(4-16)

and the first equation of (4-8) of the uncoupled system is

\[
\{X^A\}_a = [H^A]_{aa} \{F^A\}_a + [H^A]_{ac} \{F^A\}_c
\]  

(4-17)
When the systems are coupled, the force on "A" from "B" is given by \( \{ F^A \}_c \) from (4-15) and the corresponding response associated with that coupling force is \( \{ X^S \}_a \) which then becomes

\[
\{ X^S \}_a = \begin{bmatrix} H^A_{aa} \end{bmatrix} \{ F^A \}_a + \begin{bmatrix} H^A_{ac} \end{bmatrix} \{ F^A \}_c
\]

(4-18)

then (4-16) and (4-18) combine to give

\[
\begin{bmatrix} H^S_{aa} \end{bmatrix} \{ F^S \}_a + \begin{bmatrix} H^S_{ac} \end{bmatrix} \{ F^S \}_c + \begin{bmatrix} H^S_{ab} \end{bmatrix} \{ F^S \}_b
\]

\[
= \begin{bmatrix} H^A_{aa} \end{bmatrix} \{ F^A \}_a + \begin{bmatrix} H^A_{ac} \end{bmatrix} \{ F^A \}_c
\]

(4-19)
System Modeling Techniques

The FRF, \( [H^S]_{aa} \) is developed realizing that \( \{F^S\}_c \), \( \{F^S\}_b \), \( \{F^B\}_b \) are zero.

Substituting (4-15) into (4-19) and simplifying, allows for the calculation of \( [H^S]_{aa} \) in terms of the uncoupled FRF matrices as:

\[
[H^S]_{aa} = [H^A]_{aa} - [H^A]_{ac} [H^A]_{cc} + [H^B]_{cc}^{-1} [H^A]_{ca}
\]  
(4-20)
System Modeling Techniques

Schematically this is shown as

\[ h_{ij}^S = h_{ij}^A - \left[ H^A \right]_{ic} \left[ H^A \right]_{cc} + \left[ H^B \right]_{cc}^{-1} \{ H^A \}_{cj} \]

- FRFs describing output response points
- FRFs describing connection points
- FRFs describing input force points
**System Modeling Techniques**

**Additional relations are derived as**

\[
\begin{align*}
[H^S]_{ac} &= [H^A]_{ac} [H^A]_{cc} + [H^B]_{cc}^{-1} [H^B]_{cc} \\
[H^S]_{ab} &= [H^A]_{ac} [H^A]_{cc} + [H^B]_{cc}^{-1} [H^A]_{cb} \\
[H^S]_{bc} &= [H^A]_{cc} [H^A]_{cc} + [H^B]_{cc}^{-1} [H^B]_{cc} \\
[H^S]_{cb} &= [H^A]_{cc} [H^A]_{cc} + [H^B]_{cc}^{-1} [H^B]_{cb} \\
[H^S]_{bb} &= [H^B]_{bb} - [H^B]_{bc} [H^A]_{cc} + [H^B]_{cc}^{-1} [H^B]_{cb}
\end{align*}
\]
Comparison of Some System Modeling Approaches

Excerpts of results from:
System Modeling Application

MODAL TO MODAL SOLUTION

Using modal component representations, the following system models were developed:

- 5 modes Component A - 5 modes Component B
- 10 modes Component A - 10 modes Component B
- 5 modes Component A - 10 modes Component B

FULL SPACE PHYSICAL MODEL

COMPONENT A

MODAL SPACE MODEL

MODAL TIE MATRIX

COMPONENT B

FULL SPACE PHYSICAL MODEL
System Modeling Application

MODAL TO MODAL SOLUTION

The system model results were

<table>
<thead>
<tr>
<th>Mode</th>
<th>Reference</th>
<th>5A+5B</th>
<th>10A+10B</th>
<th>5A+10B</th>
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<tbody>
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<td>30.98</td>
<td>30.98</td>
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<tr>
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<td>135.87</td>
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<td>280.05</td>
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<tr>
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<tr>
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<td>526.81</td>
<td>527.40</td>
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</table>

- 5 modes yielded good results for lower modes; truncation affected higher modes
- 10 modes improved results significantly
- 5 modes for A and 10 modes for B also produced very good results
  - truncation of B is more critical than truncation of A
System Modeling Application

REDUCED TO REDUCED SOLUTION

Using a reduced physical representation for Component A and B (both Guyan and SEREP), a system model was developed using only 5 dof for each of the components.
System Modeling Application

REDUCED TO REDUCED SOLUTION - 5 GOOD DOF

The system model results were

<table>
<thead>
<tr>
<th>Mode</th>
<th>Reference</th>
<th>Guyan</th>
<th>SEREP</th>
</tr>
</thead>
<tbody>
<tr>
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<td>9.63</td>
</tr>
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<td>279.88</td>
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</table>

- Guyan reduced component loses accuracy after the second mode of the system even though this is a reasonably good selection of points
- SEREP reduced component has good results for first 6 modes of the system
- SEREP reduced models produced better results when compared to Guyan
System Modeling Application

**IMPEDANCE SOLUTION**

Using a hybrid model representation (FRFs) for Component A and B, the following system models were developed with FRFs synthesized as follows:

- 5 modes for Component A and B
- 10 modes for Component A and B
- All modes for Component A and B
System Modeling Application

**Impedance Modeling**

**FRF with 5 Modes**

**FRF with 10 Modes**

**FRF with All Modes**
System Modeling Application

Impedance Modeling

The system model results were

<table>
<thead>
<tr>
<th>Mode</th>
<th>Reference</th>
<th>5A+5B</th>
<th>10A+10B</th>
<th>All</th>
</tr>
</thead>
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<tr>
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<td>305.97</td>
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<td></td>
<td>306.78</td>
</tr>
</tbody>
</table>

The following can be seen from these results:

- as more modes are added for the synthesis the results improve
- notice that the case of 5A+5B produces the same results as the SEREP to SEREP reduced modes with 5 modes
System Modeling Application

**COMPARISON OF MODAL - SEREP REDUCED - IMPEDANCE**

The system model results for each of the different approaches using 5 modes are essentially identical.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Reference</th>
<th>Modal</th>
<th>Reduced</th>
<th>Hybrid</th>
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<td>135.83</td>
<td>135.87</td>
<td>135.87</td>
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</tbody>
</table>
Application of Hybrid Modeling Techniques for Computer Related Equipment
Excerpts of results from IMAC 15 paper

Impedance Modeling for Machine Tool Applications
Excerpts of results from IMAC 18 paper
Impedance Modeling Applications

In addition to more conventional system modeling approaches, measured frequency response functions can also be used to assemble systems and provide more realistic boundary conditions.
Impedance Modeling Applications

Use of impedance modeling techniques for development of system model using finite element component models to develop FRFs
Impedance Modeling - FEM Components

Impedance Modeling is not just for test ! ! !

- FEM models of individual components can become large
- Combining components results in very large system models which can become difficult to handle and interpret
- Impedance models are an effective approach for the development of a complete system model
Impedance Modeling - FEM Components

Impedance Model and FEM Results

- **FEM model results only obtained up to 150 Hz**
- **Impedance model results produced essentially identical results**
- **Only responses at connection points and locations of applied force and desired response are necessary to develop this model**
Impedance Modeling - FEM Components

Impedance Model allows for other than traditional FEM boundary conditions

- FEM model can only handle simplistic B.C.

- Impedance model contains the actual boundary condition of the attached component

- Overall improved system characteristics are obtained
Impedance Modeling Applications

Machine tool part dynamic characterization using measured component FRFs
Impedance Modeling - Test & FEM Components

**FEM boundary conditions for Cam Shaft in Lathe Machine Assembly difficult to model**

- **FEM model can only handle simplistic B.C.**
- **What are the proper B.C. to be used in the FEM to obtain the correct frequencies**
- **No FEM of the lathe is available - nor does anyone want to put one together**
Impedance Modeling - Test & FEM Components

Impedance information at the attachment points on the lathe and cam shaft are all that is needed

• FEM model in free-free condition used to synthesize FRFs needed

• Impedance measurements at lathe attachment points measured

• Lathe chuck measurements made with impact technique
Impedance Modeling - Test & FEM Components

Impedance Modeling results are compared to actual measured FRFs for several configurations.

![Graphs showing impedance modeling results compared to actual measurements for different configurations.](image-url)