Analytical Model Improvement & Localization Approaches

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Model Improvement Considerations

Objectives of this lecture:

• Describe the Analytical Model Improvement approach (AMI)

• Describe the Skyline Stiffness Optimization technique and the Mass and Stiffness Optimization technique (SSO/MSSO)

• A significant amount of effort is required to completely describe all aspects of these techniques
System Optimization with Phantom Elements

With a specified performance level identified, modification or adjustment of the system matrices is required to meet requirements.

Several approaches have been customized for this application.

These modified systems are then used for system disassembly to determine the cascaded component required characteristics.
Phantom elements can be used with FEM skyline topology

Alternate scenarios are needed when the component is obtained from reduced model or superelement formulation since no topology exists.
**Optimization Scenarios with Phantom Elements**

Any optimization approach can be used. The use of phantom elements is easily accomplished with FEM skyline topology.

Alternate scenarios are needed when the component is obtained from reduced model or superelement formulation since no topology exists.

Change allowed anywhere in matrix.

Connector allowed to change.

Change not allowed in white space.
Analytical Model Improvement

Equation of Motion

\[ [M]_n \{\ddot{x}(t)\}_n + [C]_n \{\dot{x}(t)\}_n + [K]_n \{x(t)\}_n = \{F(t)\}_n \]

with no damping assumed becomes

\[ [M]_n \{\ddot{x}(t)\}_n + [K]_n \{x(t)\}_n = \{F(t)\}_n \]

The eigen solution of this is

\[ ([K]_n - \lambda [M]_n) \{x\}_n = \{0\} \]

with resulting eigenvalues and eigenvectors as

\[ \{\omega\}_n, \quad [U]_n \]
Analytical Model Improvement

The modal transformation is given by

$$\{x\}_n = [U]_{nm} \{p\}_m$$

which is substituted into the equation of motion

$$[M]_n [U]_{nm} \{\ddot{p}\}_m + [K]_n [U]_{nm} \{p\}_m = \{F(t)\}_n$$

The equations are then put into normal form

$$[U]^T_{nm} [M]_n [U]_{nm} \{\ddot{p}\}_m + [U]^T_{nm} [K]_n [U]_{nm} \{p\}_m = [U]^T_{nm} \{F(t)\}_n$$

Modal Mass

$$[U]^T_{nm} [M]_n [U]_{nm} = [\bar{M}]_m = [I]_m$$

Modal Stiffness

$$[U]^T_{nm} [K]_n [U]_{nm} = [\bar{K}]_m = [\Omega^2]_m$$
Analytical Model Improvement

The reduction of large models follows the typical model reduction methodology addressed elsewhere

\[
\begin{align*}
\mathbf{M}_a &= \left[ \mathbf{T}^T_{na} \right] \mathbf{M}_n \left[ \mathbf{T} \right]_{na} \\
\mathbf{K}_a &= \left[ \mathbf{T}^T_{na} \right] \mathbf{K}_n \left[ \mathbf{T} \right]_{na}
\end{align*}
\]
Analytical Model Improvement

The Modal Assurance Criteria (MAC) and the Pseudo-Orthogonality Check (POC) are useful tools for assessment of updated models.

\[
MAC_{i,j} = \frac{[u_i]^T [e_j]^2}{[u_i]^T [u_i]} \frac{[e_j]^T [e_j]}{}
\]

\[
POC_{\text{ndof}} = [U_I]^T [M_I]_n [U_{\text{target}}]_n
\]
Analytical Model Improvement

The mass and stiffness matrices can be estimated from the direct inversion of the measured or target data of modal mass and modal stiffness

\[ [U]^T_{nm} [M]_n [U]_{nm} = [\overline{M}]_m = [I]_m \]

\[ [U]^T_{nm} [K]_n [U]_{nm} = [\overline{K}]_m = [\Omega^2]_m \]

which results in

\[ [U]^T_{nm}^g [\overline{M}]_m [U]_{nm}^g = [M]_n \]

\[ [U]_{nm}^T [\overline{K}]_m [U]_{nm}^g = [K]_n \]

but the resulting matrices are questionable.
Analytical Model Improvement (Berman & Nagy)

The discrepancy of modal mass in modal space can be written as

$$\Delta \bar{M} = [U_I]^T [M_I] [U_I] - [U_I]^T [M_S] [U_I] = [I] - [\bar{M}_S]$$

and can be projected back to physical space through a mass weighted generalize inverse

$$[\Delta M] = \left( [\bar{M}_S]^{-1} [U_I]^T [M_S]^T \right)^T [\Delta \bar{M}] \left( [\bar{M}_S]^{-1} [U_I]^T [M_S]^T \right)$$

where

$$[\Delta M] = [V]^T [\Delta \bar{M}] [V]$$  \hspace{1cm}  $$[\Delta M] = [V]^T [\Delta \bar{M}] [V]$$
Analytical Model Improvement – Mass Alone

The improved mass can then be written as

$$[M_I] = [M_S] + [V]^T [I] - [M_S] [V]$$

This mass is not necessarily related to the stiffness of the system.
Analytical Model Improvement - Summary

The improved mass is given by

\[ [M_I] = [M_S] + [V]^T ([I] - [\overline{M}_S]) [V] \]

\[ [\overline{M}_S] = [E]^T [M_S] [E] \]

Discrepancy of modal mass in modal space

\[ [\Delta M] = ([I] - [\overline{M}_S]) \]

Projection of analytical FEM seed mass from physical space to modal space

\[ [\Delta M] = [V]^T ([I] - [\overline{M}_S]) [V] \]

Projection of the discrepancy of modal mass in modal space to physical space

\[ [V] = [\overline{M}_S]^{-1} [E]^T [M_S] \]

Note: E and \( U_I \) are often used interchangeably as "target" shapes.
The discrepancy of modal stiffness in modal space can be written as

\[
\]

and can be projected back to physical space through a mass weighted generalize inverse

\[
\]

where

\[
[\Delta K] = [V]^T [\Delta \overline{K}] [V]
\]

\[
[V] = [\overline{M}_S]^{-1} [U_I]^T [M_S]
\]
Analytical Model Improvement – Stiffness Alone

The improved stiffness can then be written as

$$[K_I] = [K_S] + [V]^T [\Omega^2] - [\bar{K_S}] [V]$$

This stiffness is not necessarily related to the mass of the system.
Analytical Model Improvement

The mass and stiffness will be somewhat representative of the actual mass and stiffness since the process is “seeded” with approximate estimates of the actual system.

However, the eigensolution using these mass and stiffness matrices may not necessarily produce the exact same frequencies and mode shapes used as the target (measured) information.

\[
[M_I] = [M_S] + [V]^T [I] - [\bar{M}_S] [V]
\]

\[
[K_I] = [K_S] + [V]^T [\Omega^2] - [\bar{K}_S] [V]
\]
The force balance associated with the eigenvalue problem is given as

\[
\begin{align*}
\{\ddot{x}\}_n &= -\lambda \{x\}_n = -[U]_{nm} [\Omega^2]_{mm} \{p\}_m \\
\{x\}_n &= [U]_{nm} \{p\}_m
\end{align*}
\]

\[
[K]_n - \lambda [M]_n \{x\}_n = \{0\}
\]

and is projected to modal space as

\[
[U^T \{K\} \{U\}] = [U^T \{M\} \{U\} \Omega^2]
\]
Analytical Model Improvement (Leung/O’Callahan)

Using the weighted generalized inverse for the common form of the solution gives

\[
[K_I] = [K_S] + [V]^T [\Omega^2] + [\bar{K}_S] [V] - ([K_S] [U_I] [V]) - ([K_S] [U_I] [V])^T
\]

This uses the equations of motion as part of the process with the improved mass.

This has very special features when updating models.
Analytical Model Improvement

\[
[M_t] = [M_S] + [V]^T [I] - \lambda [M_S] [V] \\
[K_t] = [K_S] + [V]^T [\Omega^2] + [K_S] [V] - \mu [K_S] [U, [V]] - [K_S] [U, [V]]^T
\]

Finite Element

Test Results

Update Results

42.2
45.7
71.4
100.3
130.2
157.5
188.8
210.2

42.5
48.1
75.9

100.3
130.2
157.5
188.8
210.2
AMI - Matrix Smearing

System Model

System Matrix
Stiffness Skyline Optimization (SSO) - Kammer

Using the original finite element matrices as seeding matrices, the updated system characteristics can be developed from the motion of equation as

\[
[K_1 U_1 U_1^T M_1] = [M_1 U_1 \Omega^2 U_1^T M_1]
\]

A series of substitutions are needed to put the equations into a packed form with non-zero terms separated from the zero terms of the matrix.
These packed matrices are

\[
\begin{align*}
[K^I]_n & \Rightarrow \{Y^K\}_s \\
[M^I]_n[U]_{nr}[(\Omega^2)] & \Rightarrow \{D\}_s \\
[B]_s \{Y^K\}_s & = \{D\}_s
\end{align*}
\]
Example of Matrix Multiplication

Zero terms exist outside of FEM connectivity in $[M]$ & $[K]$

Matrix Multiplication Example

$$
\begin{bmatrix}
1 & -1 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 \\
0 & 0 & -1 & 2 & -1 \\
0 & 0 & 0 & -1 & 1
\end{bmatrix}
\begin{bmatrix}
y_1 \\
y_2 \\
y_3 \\
y_4 \\
y_5 
\end{bmatrix}
= \{D\}_{15}
$$

Matrix multiplication efficiency

Partitioning out zero terms

\[
\begin{align*}
1 \times y_1 &= y_1 \\
-1 \times y_2 &= -y_2 \\
0 \times y_3 &= 0 \\
0 \times y_4 &= 0 \\
0 \times y_5 &= 0
\end{align*}
\]
Mass and Stiffness Skyline Optimization (MSSO)

Using the original finite element matrices as seeding matrices, the updated system characteristics can be developed from the motion of equation as

\[
[U_I] [U_I^T] [K_I] - [U_I] [\Omega^2] [U_I^T] [M_I] = 0
\]

A series of substitutions are needed to put the equations into a packed form with non-zero terms separated from the zero terms of the matrix
Mass and Stiffness Skyline Optimization (MSSO)

These packed matrices are

\[
[U]_{nr}[U]_{nr}^T \Rightarrow [P]_{fg} \quad [U]_{nr}[[\Omega]^2]_r[U]_{nr}^T \Rightarrow [P]_{fg}
\]

\[
[K^I]_n \Rightarrow \{Y\}^K_g \quad [M^I]_n \Rightarrow \{Y\}^M_g
\]

\[
[U]_{nr}^T[M^I]_n[U]_{nr} \Rightarrow [R]_{hg} \{Y\}_g^M
\]

\[
[I]_r \Rightarrow \{V\}_h
\]
SSO and MSSO - Skyline Containment

System Model

System Matrix
SSO and MSSO - Phantom Connectivity Technique

System Model

Phantom Connectivity Technique

System Matrix