Consistent Vector Scaling

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Consistent Vector Scaling

Objectives of this lecture:

• Identify the various groups of vectors sets that may exist as part of a modal data set
• Describe the consistent vector scaling
Consistent Vector Scaling

The set of modal vectors may consist of rigid body vectors, experimentally measured vectors and supplemental analytical vectors as shown

\[ \Phi = \begin{bmatrix} R & : & E & : & A \end{bmatrix} \]
Consistent Vector Scaling

The experimental vectors may have been obtained through an expansion process using

\[
\begin{bmatrix}
E_n \\
E_d
\end{bmatrix} = \begin{bmatrix}
E_a \\
E_d
\end{bmatrix} = [T] E_a
\]

\(X_F = \) full set of dof's

\(X_A = \) active set of dof's

\[
\begin{bmatrix}
E_n \\
E_d
\end{bmatrix} = \begin{bmatrix}
E_a \\
E_d
\end{bmatrix} = [T_u] E_a = [U_n] [U_a]^g [E_a]
\]

**NOTE:** SEREP used for expansion process !!!
Consistent Vector Scaling

The improved mass is given by

\[ [M_I] = [M_S] + [V]^T[I] - [\bar{M}_S][V] \]

\[ [\bar{M}_S] = [E]^T[M_S][E] \]

Projection of analytical FEM seed mass from physical space to modal space

\[ [\Delta \bar{M}] = [I] - [\bar{M}_S] \]

Discrepancy of modal mass in modal space

\[ [\Delta M] = [V]^T[I] - [\bar{M}_S][V] \]

Projection of the discrepancy of modal mass in modal space to physical space

\[ [V] = [\bar{M}_S]^{-1}[E]^T[M_S] \]
Consistent Vector Scaling

Vector set  \( \Phi = \begin{bmatrix} R & E & A \end{bmatrix} \) satisfy orthogonality

\[
[\Phi]^T [M][\Phi] = [I]
\]

\[
[\Phi]^T [M][\Phi] = \begin{bmatrix}
\end{bmatrix}
\]
**Consistent Vector Scaling**

\[ \Phi = \begin{bmatrix} R & E & A \end{bmatrix} \]

**Vector set**

\[ [E]^T [M][E] = [I] \]

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**Partition #1**

\[ [E]^T M E = I \]

Since the mass was improved based on the expanded experimental vectors, the relationship must be satisfied.

Expanding this equation out gives

\[ [E]^T M E = [E]^T \left[ \begin{bmatrix} E & S \end{bmatrix} + [V]^T (I - [E]^T S E) V \right] E \]


and recalling that

\[ [V] E = \begin{bmatrix} E & S \end{bmatrix}^{-1} E S E = I \]

and substituting equation (8) into equation (7) gives


as expected.
**Consistent Vector Scaling**

\[ \Phi = \begin{bmatrix} R & E & A \end{bmatrix} \]

**Vector set**

\[ [R]^T [M][R] = [I] \]

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**Partition #2**

\[ R^T M R = I \]

It is not obvious that the rigid body modes and the improved mass will satisfy

\[ R^T M R = I \]

(9)

Let's look at expanding equation (9)

\[ R^T M R = R^T \left[ M_s + V^T (I - E^T M_s E) V \right] R \]

(10)

\[ = R^T M_s R + (V R)^T (I - E^T M_s E) (V R) \]

and note that we can write

\[ V R = R^{-1} E^T M_s R = 0 \]

(11)

since

\[ E^T M_s R = E_a U_g^T U_n^T M_s R = 0 \]

because

\[ U_n^T M_s R = 0 \]

from the orthogonality condition.

Now equation (10) becomes

\[ R^T M R = R^T M_s R = I \]

(12)

and satisfies our requirement.
**Consistent Vector Scaling**

\[ \Phi = [R \quad E \quad A] \]

**Vector set**

\[ [R]^T [M] [E] = [0] \]

**Partition #3**

\[
\begin{align*}
R^T M E &= 0 \\
&= [R^T] [M] [E] = 0
\end{align*}
\]

We now must check the cross orthogonality relationship between the rigid body modes and the expanded experimental modes with the improved mass matrix to assure that

\[
R^T M E = 0 \quad (13)
\]

Expanding equation (13) gives

\[
R^T M E = R^T \left( M_s + V^T (I - E^T M_s E) V \right) E
\]

\[
= R^T M_s E + (V R)^T (I - E^T M_s E) (V E)
\]

From equation (11), this reduces to

\[
R^T M E = R^T M_s E \quad (15)
\]

which can be expanded to

\[
R^T M E = R^T M_s \left( U \ U^T \ E \right) = 0
\]

due to the orthogonality condition and satisfies our condition.
Consistent Vector Scaling

\[ \Phi = [ R : E : A ] \]

Vector set

\[
\begin{bmatrix} R \end{bmatrix}^T \begin{bmatrix} M \end{bmatrix} \begin{bmatrix} A \end{bmatrix} = [0]
\]

Partition #4

\[ R^T M A = 0 \]

We now must check the cross orthogonality relationship between the rigid body modes and the analytical modes with the improved mass matrix to assure that

\[ R^T M A = 0 \]  \hspace{1cm} (16) \]

Expanding equation (16) gives

\[
R^T M A = R^T \left[ M_S + V^T \left( I - E^T M_S E \right) V \right] A
\]

\[
= R^T M_S A + \left( V R \right)^T \left( I - E^T M_S E \right) \left( V A \right)
\]

From equation (11), this reduces to

\[ R^T M A = R^T M_S A = 0 \]  \hspace{1cm} (18) \]

due to the orthogonality condition and satisfies our condition.
Consistent Vector Scaling

\[ \Phi = [R \quad E \quad A] \]

Vector set

\[ [E]^T[M][A] = [0] \]

Partition #5

\[ E^T M = 0 \]

We now must check the cross orthogonality relationship between the expanded experimental modes and the analytical modes with the improved mass matrix to assure that

\[ E^T M A = 0 \]

Expanding equation (19) gives

\[ E^T M A = E^T M S A + (V E)^T (I - E^T M S E) (V A) \]

and using the same development as in equation (11), it can be shown that

\[ V A = 0 \]

which reduces equation (19) to

\[ E^T M A = E^T M S A \]

which can be expanded to give

\[ E^T M A = E^T U_0^T U_0 U^T M A = 0 \]

due to the orthogonality condition and satisfies our condition.
**Consistent Vector Scaling**

\[ \Phi = \begin{bmatrix} R & : & E & : & A \end{bmatrix} \]

**Vector set**

\[ [A]^T[M][A] = [I] \]

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**Partition #6**

It is not obvious that the analytical FEM modes and the improved mass will satisfy

\[ A^T M A = I \]  \hspace{1cm} (22) \]

Let's look at expanding equation (22)

\[ A^T M A = A^T \begin{bmatrix} M_S & V^T \left( I - E^T M_S E \right) V \end{bmatrix} A \]  \hspace{1cm} (23) \]

\[ = A^T M_S A + (V A)^T \left( I - E^T M_S E \right) (V A) \]

and note that we can write

\[ V A = M_S^{-1} E^T M_S A = 0 \]  \hspace{1cm} (24) \]

since

\[ E^T M_S A = E_a U_a^T U_n M_S A = 0 \]

because

\[ U_n M_S A = 0 \]

from the orthogonality condition.

Now equation (23) becomes

\[ A^T M A = A^T M_S A = I \]  \hspace{1cm} (25) \]

and satisfies our requirement.
Consistent Vector Scaling

\[ \Phi = [ R \quad E \quad A ] \]

Vector set satisfies orthogonality

\[ [\Phi]^T [M][\Phi] = [I] \]

\[
[\Phi]^T[M][\Phi] = \begin{bmatrix}
\end{bmatrix}
\]