Structural Dynamic Modeling Techniques & Modal Analysis Methods

System Modeling Concepts

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System Modeling Concepts

System models are generated from component models for a variety of applications

Modal Methods

Component Mode Synthesis

Frequency Based Substructuring
Modal to Modal with Tie Matrix

For component $\alpha$ (modal component):

$$\begin{bmatrix}
\omega^2 \\
U \alpha \\
p \alpha
\end{bmatrix}$$

For component $\beta$ (modal component):

$$\begin{bmatrix}
\omega^2 \\
U \beta \\
p \beta
\end{bmatrix}$$
Modal to Modal with Tie Matrix

Uncoupled Systems

\[
\begin{bmatrix}
[I]^\alpha & 0 \\
0 & [I]^\beta
\end{bmatrix}
\begin{bmatrix}
\dot{p}^\alpha \\
\dot{p}^\beta
\end{bmatrix}
+ \begin{bmatrix}
\Omega^2 \alpha & 0 \\
0 & \Omega^2 \beta
\end{bmatrix}
\begin{bmatrix}
p^\alpha \\
p^\beta
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

Component \(\alpha\)

\[
\begin{bmatrix}
\omega^2 \alpha & [U]^\alpha & p^\alpha \\
\end{bmatrix}
\]

Component \(\beta\)

\[
\begin{bmatrix}
\omega^2 \beta & [U]^\beta & p^\beta \\
\end{bmatrix}
\]
Modal to Modal with Tie Matrix

The two systems are connected with tie matrices

\[
\begin{bmatrix}
\Delta M_T \\
\Delta K_T
\end{bmatrix}
\]

The tie matrix can be projected from physical to modal space using

\[
\begin{bmatrix}
\Delta M_T \\
\Delta K_T
\end{bmatrix} = [U]^T \begin{bmatrix}
\Delta M_T \\
\Delta K_T
\end{bmatrix} [U]
\]

with

\[
[U] = \begin{bmatrix}
[U]^\alpha & 0 \\
0 & [U]^\beta
\end{bmatrix}
\]
**Modal to Modal with Tie Matrix**

Adding this to the uncoupled equations gives

\[
\begin{bmatrix}
[I]^{\alpha} & 0 \\
0 & [I]^\beta \\
\end{bmatrix} + [\Delta \bar{M}_T] \begin{Bmatrix}
\dot{p}^{\alpha} \\
\dot{p}^\beta \\
\end{Bmatrix} + \begin{bmatrix}
0 \\
[\Omega^2]^{\alpha} \\
[\Omega^2]^\beta \\
\end{bmatrix} + [\Delta \bar{K}_T] \begin{Bmatrix}
p^{\alpha} \\
p^\beta \\
\end{Bmatrix} = \begin{Bmatrix}
0 \\
0 \\
\end{Bmatrix}
\]

The eigensolution of this gives the new system frequencies and mode shapes.
Modal to Physical with Constraint

Component $\alpha$ (physical) can be partitioned into connection DOF ($c$) and other or interior DOF ($i$)

$$
\begin{align*}
\mathbf{x} &= \begin{cases} 
\mathbf{x}_c \\
\mathbf{x}_i 
\end{cases} \\
\mathbf{M}^\alpha &= \begin{bmatrix} 
\mathbf{M}_{cc} & \mathbf{M}_{ci} \\
\mathbf{M}_{ic} & \mathbf{M}_{ii} 
\end{bmatrix} \\
\mathbf{K}^\alpha &= \begin{bmatrix} 
\mathbf{K}_{cc} & \mathbf{K}_{ci} \\
\mathbf{K}_{ic} & \mathbf{K}_{ii} 
\end{bmatrix}
\end{align*}
$$

Component $\beta$ (modal) only considers connection DOF

$$
\begin{bmatrix} 
\omega^2 \\
\mathbf{U} \\
p
\end{bmatrix}^\beta
$$
Modal to Physical with Constraint

The constraint equation is given by

\[ x_c = U_c p^\beta \]

and the uncoupled equations are given as

\[
\begin{bmatrix}
[M_{cc}] & [M_{ci}]
\end{bmatrix}
\begin{bmatrix}
\ddot{x}_c \\
\ddot{x}_i \\
\dot{p}^\beta
\end{bmatrix}
+ \begin{bmatrix}
[K_{cc}] & [K_{ci}]
[K_{ic}] & [K_{ii}]
\end{bmatrix}
\begin{bmatrix}
x_c \\
x_i \\
p^\beta
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

and the relationship of constraint is

\[
\begin{bmatrix}
x_c \\
x_i \\
p^\beta
\end{bmatrix}
= \begin{bmatrix}
0 & [U_c]
[I] & 0
0 & [I]
\end{bmatrix}
\begin{bmatrix}
x_i \\
p
\end{bmatrix}
= [T]\begin{bmatrix}
x_i \\
p
\end{bmatrix}
\]
Modal to Physical with Constraint

Substituting and putting into normal form gives

\[
[T]^{T} [M]^{*}[T] \begin{bmatrix} \ddot{x}_i \\ \ddot{p}_\beta \end{bmatrix} + [T]^{T} [K]^{*}[T] \begin{bmatrix} x_i \\ p_\beta \end{bmatrix} = 0
\]

or

\[
[T]^{T} \begin{bmatrix} [M_{cc}] & [M_{ci}] \\ [M_{ic}] & [M_{ii}] \end{bmatrix} \begin{bmatrix} \ddot{x}_i \\ \ddot{p}_\beta \end{bmatrix} + [T]^{T} \begin{bmatrix} [K_{cc}] & [K_{ci}] \\ [K_{ic}] & [K_{ii}] \end{bmatrix} \begin{bmatrix} x_i \\ p_\beta \end{bmatrix} = 0
\]
Modal to Modal with Physical System Added

Component $\alpha$ and $\beta$
(modal components):

\[
\begin{bmatrix}
\omega^2 \\
U\end{bmatrix}^\alpha \quad \begin{bmatrix}
\omega^2 \\
U\end{bmatrix}^\beta
\]

$p^\alpha \quad p^\beta$

Component $\gamma$
(physical component)
only connection DOF:

\[
x = \begin{bmatrix} x_c \\ x_i \end{bmatrix}
\]

\[
[M] = \begin{bmatrix} [M_{cc}] & [M_{ci}] \\ [M_{ic}] & [M_{ii}] \end{bmatrix} \quad [K] = \begin{bmatrix} [K_{cc}] & [K_{ci}] \\ [K_{ic}] & [K_{ii}] \end{bmatrix}
\]
Modal to Modal with Physical System Added

Uncoupled system

\[
\begin{bmatrix}
[M]
\end{bmatrix}
\begin{bmatrix}
[M]^\alpha

[M]^\beta
\end{bmatrix}
\begin{bmatrix}
\ddot{x}_c

\ddot{p}^\alpha

\ddot{p}^\beta
\end{bmatrix}
\]

\[
+ \begin{bmatrix}
[K]
\end{bmatrix}
\begin{bmatrix}
[K]^\alpha

[K]^\beta
\end{bmatrix}
\begin{bmatrix}
\dot{x}_c

p^\alpha

p^\beta
\end{bmatrix} = 0
\]

Constraint relation is

\[
\begin{bmatrix}
x^\alpha_c

x^\beta_c
\end{bmatrix} = \begin{bmatrix}
U^\alpha_c

U^\beta_c
\end{bmatrix}
\begin{bmatrix}
p^\alpha

p^\beta
\end{bmatrix} = [t]p
\]
Modal to Modal with Physical System Added

For all DOF

\[
\begin{align*}
\begin{bmatrix} x^\alpha_c \\ x^\beta_c \\ p^\alpha \\ p^\beta \end{bmatrix} &= \begin{bmatrix} U^\alpha_c & 0 \\ 0 & U^\beta_c \\ I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} p^\alpha \\ p^\beta \end{bmatrix} = [T]p = \begin{bmatrix} p^\alpha \\ p^\beta \end{bmatrix} \\
\{x_c, p^\alpha, p^\beta\} &= \begin{bmatrix} x_c \\ p^\alpha \\ p^\beta \end{bmatrix} = \begin{bmatrix} [t]^\alpha & [t]^\beta \\ I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} p^\alpha \\ p^\beta \end{bmatrix} = [T]p^* 
\end{align*}
\]
Modal to Modal with Physical System Added

This leads to

\[
[T]^T [M]^* [T] \begin{bmatrix} \ddot{X}_c \\ \dot{p} \end{bmatrix} + [T]^T [K]^* [T] \begin{bmatrix} X_c \\ p \end{bmatrix} = 0
\]

where

\[
[K]^{**} = [T]^T [K]^* [T] = \begin{bmatrix} [K]^\alpha + [t]^\alpha^T [K] [t]^\alpha & [t]^\alpha^T [K] [t]^\beta \\ [t]^\beta^T [K] [t]^\alpha & [K]^\beta + [t]^\beta^T [K] [t]^\beta \end{bmatrix}
\]

\[
[M]^{**} = [T]^T [M]^* [T] = \begin{bmatrix} [M]^\alpha + [t]^\alpha^T [M] [t]^\alpha & [t]^\alpha^T [M] [t]^\beta \\ [t]^\beta^T [M] [t]^\alpha & [M]^\beta + [t]^\beta^T [M] [t]^\beta \end{bmatrix}
\]
Physical System with Modal Components Added

Component $\alpha$ and $\beta$
(modal components):

$$
\begin{bmatrix}
\omega^2 \\
U \\
p
\end{bmatrix}^\alpha
\begin{bmatrix}
\omega^2 \\
U \\
p
\end{bmatrix}^\beta
$$

Component $\gamma$
(physical component)
only connection DOF:

$$
\begin{bmatrix}
M \\
K
\end{bmatrix}
= 
\begin{bmatrix}
M_{cc} & M_{ci} \\
M_{ic} & M_{ii}
\end{bmatrix}
\begin{bmatrix}
K_{cc} & K_{ci} \\
K_{ic} & K_{ii}
\end{bmatrix}
$$

$$
x = \begin{bmatrix} x_c \\ x_i \end{bmatrix}
$$
Physical System with Modal Components Added

Uncoupled system
\[
\begin{bmatrix}
[M_{cc}] & [M_{ci}] \\
[M_{ic}] & [M_{ii}]
\end{bmatrix}
\begin{bmatrix}
\ddot{x}_c \\
\ddot{x}_i
\end{bmatrix}
+ \begin{bmatrix}
[K_{cc}] & [K_{ci}] \\
[K_{ic}] & [K_{ii}]
\end{bmatrix}
\begin{bmatrix}
x_c \\
x_i
\end{bmatrix}
= \begin{bmatrix}
[M]^{\alpha} \\
[K]^{\alpha}
\end{bmatrix} \begin{bmatrix}
\ddot{p}^{\alpha} \\
p^{\alpha}
\end{bmatrix}
+ \begin{bmatrix}
[M]^{\beta} \\
[K]^{\beta}
\end{bmatrix} \begin{bmatrix}
\ddot{p}^{\beta} \\
p^{\beta}
\end{bmatrix}
\]

Constraint relation is
\[
x_c = \begin{bmatrix}
[U_c]^{\alpha} & [U_c]^{\beta}
\end{bmatrix} \begin{bmatrix}
p^{\alpha} \\
p^{\beta}
\end{bmatrix}
\]
For all DOF

\[
\begin{align*}
\begin{bmatrix} x_c \\ x_i \\ p^\alpha \\ p^\beta \end{bmatrix} &= \begin{bmatrix} 0 & \mathbf{U}_c^\alpha & \mathbf{U}_c^\beta \\ \mathbf{I} & \mathbf{I} & \mathbf{I} \end{bmatrix} \begin{bmatrix} x_i \\ p^\alpha \\ p^\beta \end{bmatrix}
\end{align*}
\]
**Physical System Modal Components Added**

This then becomes

\[
[T]^T [M]^* [T] \begin{bmatrix}
\dot{x}_i \\
\ddot{p}^\alpha
\end{bmatrix} + [T]^T [K]^* [T] \begin{bmatrix}
x_i \\
p^\alpha
\end{bmatrix} = 0
\]

where

\[
[M]^{**} = [T]^T [M]^* [T] = \begin{bmatrix}
[M_{ii}] & [M_{ic}] [U_c]^\alpha \\
[U_c]^\alpha^T [M_{ci}] & [M_{cc}] [U_c]^\alpha + [M]^\alpha & [M_{ic}] [U_c]^\beta \\
[U_c]^\beta^T [M_{ci}] & [U_c]^\beta^T [M_{cc}] [U_c]^\alpha & [U_c]^\beta^T [M_{cc}] [U_c]^\beta + [M]^\beta
\end{bmatrix}
\]

\[
[K]^{**} = [T]^T [K]^* [T] = \begin{bmatrix}
[K_{ii}] & [K_{ic}] [U_c]^\alpha \\
[U_c]^\alpha^T [K_{ci}] & [K_{cc}] [U_c]^\alpha + [K]^\alpha & [K_{ic}] [U_c]^\beta \\
[U_c]^\beta^T [K_{ci}] & [U_c]^\beta^T [K_{cc}] [U_c]^\alpha & [U_c]^\beta^T [K_{cc}] [U_c]^\beta + [K]^\beta
\end{bmatrix}
\]
Component Mode Synthesis

Two components: $\alpha$ and $\beta$

For each component, the equation of motion can be written in partitioned form, with

$\begin{bmatrix} M_{cc} & M_{ci} \\ M_{ic} & M_{ii} \end{bmatrix} \begin{bmatrix} \ddot{x}_c \\ \ddot{x}_i \end{bmatrix} + \begin{bmatrix} K_{cc} & K_{ci} \\ K_{ic} & K_{ii} \end{bmatrix} \begin{bmatrix} x_c \\ x_i \end{bmatrix} = \begin{bmatrix} f_c \\ f_i \end{bmatrix}$

$\dot{x}_c = \text{juncture coordinates}$

$\dot{x}_i = \text{interior coordinates}$
Component Mode Synthesis

We can represent the physical coordinates $x$ in terms of component generalized coordinates $p$:

$$x = \Psi p$$

where $\Psi$ = a matrix of component modes.

For the Craig-Bampton method, constraint modes and fixed-interface normal modes are used as component modes.
Component Mode Synthesis

Fixed-Interface normal modes

$U_m$ (where “m” refers to kept modes)

All juncture coordinates are constrained

Obtain the normal modes by solving the eigenvalue problem:

$$\left(k - \omega^2 m\right)x = 0$$

It is assumed that the modes are scaled to unit modal mass.
Component Mode Synthesis

Constraint modes

Partition physical coordinates $x$ into two sets: $c$, the constrained coordinates—the coordinates relative to which the constraint modes will be defined; and $i$, the remaining (interior) coordinates

Statically impose a unit displacement on one constrained coordinate, and maintain a zero displacement on the other constrained coordinates. The remaining $i$ coordinates are free to deform.
Component Mode Synthesis

Constraint modes

The constraint mode matrix is therefore

\[ T^c = \begin{bmatrix} I_{cc} \\ C_{ic} \end{bmatrix} = \begin{bmatrix} I_{cc} \\ -K_{ii}^{-1}K_{ic} \end{bmatrix} \]

Where \( T^c = \text{constraint mode transformation} \)
Component Mode Synthesis

Applying the Craig Bampton method:

Let \( x = U_m p_m + C p_c \) \( \text{for each component, where} \)
\( U_m = \text{kept fixed-interface normal modes, and} \)
\( C = \text{constraint modes.} \)
Component Mode Synthesis

Since the normal modes are fixed-interface modes, and \( p_c \equiv x_c \), this can be written in partitioned form as

\[
\begin{bmatrix}
\mathbf{x}_c \\
\mathbf{x}_i
\end{bmatrix} =
\begin{bmatrix}
\mathbf{I}_{cc} & \mathbf{0} \\
\mathbf{C}_{ic} & \mathbf{U}_{im}
\end{bmatrix}
\begin{bmatrix}
p_c \\
p_m
\end{bmatrix}
\]

where

\( \mathbf{I}_{cc} = \) identity associated with unit displacement at connection DOF (for constraint modes)

\( \mathbf{C}_{ic} = \) resulting constraint modes, equal to \( \mathbf{K}_{ii}^{-1} \mathbf{K}_{ic} \)

\( \mathbf{U}_{im} = \) normal modes of system with \( c \) constraints applied (fixed-interface normal modes)

Note: \( n - c = i \)

\( n = \) all points (as usual)

\( c = \) constrained points

\( i = \) interior points
**Component Mode Synthesis**

In general, for each component, the mass and stiffness matrices are

\[
\mu = \Psi^T M \Psi \\
\kappa = \Psi^T K \Psi
\]

where $\Psi$ is the matrix of component modes.
Component Mode Synthesis

Using the Craig Bampton method, the mass is given by

\[
\mu = \begin{bmatrix}
\mu_{cc} & \mu_{cm} \\
\mu_{mc} & \mu_{mm}
\end{bmatrix}
\]

where

\[
\mu_{mm} = I_{mm}
\]

\[
\mu_{mc} = \mu_{cm} = U_{im}^T (M_{ii}C_{ic} + m_{ij})
\]

\[
\mu_{cc} = C_{ic}^T (m_{ii}C_{ic} + m_{ic}) + m_{ci}C_{ic} + m_{cc}
\]
Component Mode Synthesis

Using the Craig Bampton method, the stiffness is given by

\[
\kappa = \begin{bmatrix}
\kappa_{cc} & \kappa_{cm} \\
\kappa_{mc} & \kappa_{mm}
\end{bmatrix}
\]

Where

\[
\kappa_{mm} = \Lambda_{mm} = \Omega^2
\]

\[
\kappa_{mc}^T = \kappa_{cm} = 0
\]

\[
\kappa_{cc} = K_{cc} - K_{ci}K_{ii}^{-1}K_{ic}
\]

Note that \(\kappa_{cc}\) is the Guyan reduced stiffness matrix
Component Mode Synthesis

To assemble system matrices, first write the equation of interface displacement compatibility (with \( p_c^\alpha \) being the dependent coordinate):

\[
p_c^\beta - p_c^\alpha = 0
\]

Written in the form

\[
Dp = 0
\]

this is

\[
\begin{bmatrix}
-1 & 0 & I & 0
\end{bmatrix}
\begin{bmatrix}
p_c^\alpha \\
p_m^\alpha \\
p_m^\beta \\
p_c^\beta \\
p_m^\beta 
\end{bmatrix} = 0
\]

therefore

\[
D = \begin{bmatrix}
-1 & 0 & I & 0
\end{bmatrix}
\]

\[
D_{DD} = -I
\]

\[
D_{DI} = \begin{bmatrix}
0 & I & 0
\end{bmatrix}
\]

\( D = \) dependent

\( I = \) linearly independent
Component Mode Synthesis

Then define \( S \) so \( p = S q \), where 
\( p = \) original generalized coordinates, and 
\( q = \) new generalized coordinates.

In this case 
\[
\begin{bmatrix}
  p^\alpha_c \\
  p^\alpha_m \\
  p^\beta_c \\
  p^\beta_m
\end{bmatrix} =
\begin{bmatrix}
  1 & 0 & 0 \\
  0 & 1 & 0 \\
  1 & 0 & 0 \\
  0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
  p^\beta_c \\
  p^\beta_m \\
  p^\alpha_c \\
  p^\alpha_m
\end{bmatrix}
\]
**Component Mode Synthesis**

**To get synthesized system $M$ and $K$ matrices:**

$$M^{\alpha\beta} = S^T \mu S$$

$$K^{\alpha\beta} = S^T \kappa S$$

$$M^{\alpha\beta} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1
\end{bmatrix} \begin{bmatrix}
\mu_{cc} & \mu_{cm} & 0 & 0 \\
\mu_{mc} & \mu_{mm} & 0 & 0 \\
0 & 0 & \mu_{cc} & \mu_{cm} \\
0 & 0 & \mu_{mc} & \mu_{mm}
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0 \\
0 & 1 & 0
\end{bmatrix}$$
System Modeling Concepts

Then

\[
M^{\alpha \beta} = \begin{bmatrix}
M_{cc} & M_\alpha & M_\beta \\
M_\alpha & M_{mm} & 0 \\
M_\beta & 0 & M_{mm}
\end{bmatrix}
\]

where

\[
M_\alpha = I_\alpha
\]

\[
M_\beta = I_\beta
\]

\[
M_\alpha = \left( M_\alpha \right)^T = \mu_\alpha
\]

\[
M_\beta = \left( M_\beta \right)^T = \mu_\beta
\]

\[
M_{cc} = \mu_\alpha + \mu_\beta
\]
System Modeling Concepts

And

\[ K^{\alpha\beta} = \begin{bmatrix} K_{cc} & 0 & 0 \\ 0 & K_{mm} & 0 \\ 0 & 0 & K_{kk}^{\beta} \end{bmatrix} \]

Where

\[ K_{mm}^{\alpha} = \Lambda_{mm}^{\alpha} \quad = \text{diagonal matrix, modal stiffness of} \ \alpha \]

\[ K_{mm}^{\beta} = \Lambda_{mm}^{\beta} \quad = \text{diagonal matrix, modal stiffness of} \ \beta \]

\[ K_{cc} = \kappa_{cc}^{\alpha} + \kappa_{cc}^{\beta} \quad = \text{full matrix of stiffness terms for reduced} \ \alpha \ \text{and} \ \beta \]
Then solve the equation of motion for the assembled system:

\[ M^{\alpha\beta} \ddot{q} + K^{\alpha\beta} q = 0 \]

Transform from \( q \) to \( p \) coordinates using

\[ p = S q \]

and then from \( p \) to \( u \) (physical coordinates) using

\[
\begin{bmatrix}
\dot{\mathbf{x}}_c \\
\dot{\mathbf{x}}_i
\end{bmatrix} =
\begin{bmatrix}
I_{cc} & 0 \\
C_{ic} & U_{im}
\end{bmatrix}
\begin{bmatrix}
p_c \\
p_m
\end{bmatrix}
\]
Impedance Modeling Techniques

Consider a cantilever beam. It is desired to estimate the FRF between point c and b when the tip of the beam is pinned to ground.

In particular, the FRF $h_{cb}$ when $x_a = 0$
Impedance Modeling Techniques

The response at "a" is related to the force at "a" and "b" through

\[ x_a = h_{ab} f_b + h_{aa} f_a \]

where \( x_a \) is the vertical translation at the tip of the beam.

With the constraint \( x_a = 0 \), the force at point "a" becomes

\[ f_a = -h_{aa}^{-1} h_{ab} f_b \]
Impedance Modeling Techniques

The response at "c" due to an excitation at "a" and "b" is

\[ x_c = h_{ca} f_a + h_{cb} f_b \]

In order to include the effects of the constraint at "a", the force at point "a" with the constraint \( x_a = 0 \), changes this equation to

\[ \tilde{h}_{cb} = \frac{x_c}{f_b} = h_{cb} - h_{ca} h_{aa}^{-1} h_{ab} \]

which are obtained from the unconstrained system
Summary of Impedance Modeling

Frequency Response Functions can also be used to investigate structural modifications. The FRF can be written as

$$H_{ij}(j\omega) = \sum_{k=1}^{m} \frac{q_{k}u_{ik}u_{jk}}{(j\omega - p_{k})} + \frac{q_{k}u_{ik}^{*}u_{jk}^{*}}{(j\omega - p_{k}^{*})}$$

Using force balance and compatibility equations, the effects of a modification can be written in terms of the unmodified system as

$$x_{a} = H_{ab}F_{b} + H_{aa}F_{a}$$

$$F_{a} = -H_{aa}^{-1}H_{ab}F_{b}$$

$$x_{c} = H_{ca}F_{a} + H_{cb}F_{b}$$

\[\tilde{H}_{cb} = \frac{x_{c}}{F_{b}} = H_{cb} - H_{ca}H_{aa}^{-1}H_{ab}\]
Frequency Based System Modeling Techniques

Consider combining two systems together

COMPONENT (A)
- a-DOFs
- c-DOFs

SYSTEM (S)

COMPONENT (B)
- c-DOFs
- b-DOFs
The equation of motion for each component is

\[ \{X\}_n = [H]_{nn} \{F\}_n \]

where

\[ n = a + c \] for component (A)
\[ n = b + c \] for component (B).

Note that the number of "c" coordinates are the same on component (A) and (B).
FBS Modeling Techniques

**Component A can be partitioned as**

\[
\begin{align*}
\{X_A^A\}_{n} &= \begin{bmatrix} H^A_{aa} & H^A_{ac} \\ H^A_{ca} & H^A_{cc} \end{bmatrix}_{nn} \{F_A^A\}_{n} \\
\{X_A^c\}_{n} &= \begin{bmatrix} F_A^A \end{bmatrix}_{n}
\end{align*}
\]

**Component B can be partitioned as**

\[
\begin{align*}
\{X_B^B\}_{n} &= \begin{bmatrix} H^B_{cc} & H^B_{cb} \\ H^B_{bc} & H^B_{bb} \end{bmatrix}_{nn} \{F_B^B\}_{n} \\
\{X_B^c\}_{n} &= \begin{bmatrix} F_B^B \end{bmatrix}_{n}
\end{align*}
\]
FBS Modeling Techniques

When rigidly connecting Component A to Component B, compatibility implies that

\[ \{X^A\}_c = \{X^B\}_c = \{X^S\}_c \]  \hspace{1cm} (4-10)

and equilibrium at the "c" DOFs requires that

\[ \{F^A\}_c + \{F^B\}_c = \{F^S\}_c \]  \hspace{1cm} (4-11)

where "S" superscript is used to represent system comprised of Component A rigidly coupled to Component B at the connection DOFs "c"
**FBS Modeling Techniques**

The FRFs of the uncoupled system can be defined as

\[
\begin{bmatrix}
\{X^S\}_a \\
\{X^S\}_b \\
\{X^S\}_c
\end{bmatrix}_n =
\begin{bmatrix}
H^S_{aa} & H^S_{ac} & H^S_{ab} \\
H^S_{ca} & H^S_{cc} & H^S_{cb} \\
H^S_{ba} & H^S_{bc} & H^S_{bb}
\end{bmatrix}_n
\begin{bmatrix}
\{F^S\}_a \\
\{F^S\}_b \\
\{F^S\}_c
\end{bmatrix}
\]

(4-12)

From the partitioned equations for Component A and Component B, the connection DOF are

\[
\begin{align*}
\{X^A\}_c &= [H^A]_{ca} \{F^A\}_a + [H^A]_{cc} \{F^A\}_c \\
\{X^B\}_c &= [H^B]_{cc} \{F^B\}_c + [H^B]_{cb} \{F^B\}_b
\end{align*}
\]

(4-13) (4-14)
FBS Modeling Techniques

These two equations can be equated and used to solve for the connection force as

\[
\{\tilde{F}_A^A\}_c = \left[\begin{bmatrix} H^A \end{bmatrix}_{cc} + \left[\begin{bmatrix} H^B \end{bmatrix}_{cc} \right]^{-1} \left[\begin{bmatrix} H^B \end{bmatrix}_{cb} \right] \{F^B\}_b \right. - \left[\begin{bmatrix} H^A \end{bmatrix}_{ca} \right] \{F^A\}_a + \left[\begin{bmatrix} H^B \end{bmatrix}_{cc} \right] \{F^S\}_c
\]

From these equations derived above, the coupled system FRFs can be determined in terms of the uncoupled FRFs of the individual components. Equations (4-8), (4-9), (4-12) and (4-15) are used in the development of the coupled system.
FBS Modeling Techniques

As an example, \( [H^S]_{aa} \) will be derived.

The first equation of (4-12) of the coupled system is

\[
\{X^S\}_a = [H^S]_{aa} \{F^S\}_a + [H^S]_{ac} \{F^S\}_c + [H^S]_{ab} \{F^S\}_b
\]  

(4-16)

and the first equation of (4-8) of the uncoupled system is

\[
\{X^A\}_a = [H^A]_{aa} \{F^A\}_a + [H^A]_{ac} \{F^A\}_c
\]  

(4-17)
FBS Modeling Techniques

When the systems are coupled, the force on "A" from "B" is given by \( \begin{bmatrix} F^A \end{bmatrix}_c \) from (4-15)

and the corresponding response associated with that coupling force is \( \begin{bmatrix} X^S \end{bmatrix}_a \) which then becomes

\[
\begin{bmatrix} X^S \end{bmatrix}_a = \begin{bmatrix} H^A \end{bmatrix}_{aa} \begin{bmatrix} F^A \end{bmatrix}_a + \begin{bmatrix} H^A \end{bmatrix}_{ac} \begin{bmatrix} F^A \end{bmatrix}_c
\]
(4-18)

then (4-16) and (4-18) combine to give

\[
\begin{bmatrix} H^S \end{bmatrix}_{aa} \begin{bmatrix} F^S \end{bmatrix}_a + \begin{bmatrix} H^S \end{bmatrix}_{ac} \begin{bmatrix} F^S \end{bmatrix}_c + \begin{bmatrix} H^S \end{bmatrix}_{ab} \begin{bmatrix} F^S \end{bmatrix}_b
\]
\[
= \begin{bmatrix} H^A \end{bmatrix}_{aa} \begin{bmatrix} F^A \end{bmatrix}_a + \begin{bmatrix} H^A \end{bmatrix}_{ac} \begin{bmatrix} F^A \end{bmatrix}_c
\]
(4-19)
FBS Modeling Techniques

The FRF, \( [H^S]_{aa} \) is developed realizing that \( \{F^S\}_c \) \( \{F^S\}_b \) \( \{F^B\}_b \) are zero.

Substituting (4-15) into (4-19) and simplifying, allows for the calculation of \( [H^S]_{aa} \) in terms of the uncoupled FRF matrices as

\[
[H^S]_{aa} = [H^A]_{aa} - [H^A]_{ac} [H^A]_{cc} + [H^B]_{cc}^{-1} [H^A]_{ca}
\] (4-20)
Schematically this is shown as

\[ h^S_{ij} = h^A_{ij} - \left[ H^A \right]_{ic} \left[ H^A \right]_{cc} + \left[ H^B \right]_{cc}^{-1} \{ H^A \}_{cj} \]

- FRFs describing output response points
- FRFs describing connection points
- FRFs describing input force points
**FBS Modeling Techniques**

*Additional relations are derived as*

\[
[H^S]_{ac} = [H^A]_{ac} [H^A]_{cc} + [H^B]_{cc}^{-1} [H^B]_{cc}
\]

\[
[H^S]_{ab} = [H^A]_{ac} [H^A]_{cc} + [H^B]_{cc}^{-1} [H^A]_{cb}
\]

\[
[H^S]_{cc} = [H^A]_{cc} [H^A]_{cc} + [H^B]_{cc}^{-1} [H^B]_{cc}
\]

\[
[H^S]_{cb} = [H^A]_{cc} [H^A]_{cc} + [H^B]_{cc}^{-1} [H^B]_{cb}
\]

\[
[H^S]_{bb} = [H^B]_{bb} - [H^B]_{bc} [H^A]_{cc} + [H^B]_{cc}^{-1} [H^B]_{cb}
\]