Impulse Response of Second-Order Systems

INTRODUCTION

This document discusses the response of a second-order system, like the mass-spring-dashpot system shown in Fig. 1, to an impulse.

![Second-order mass-spring-dashpot system](image)

Fig. 1. Second-order mass-spring-dashpot system.

IMPULSE

An impulse is a large force applied over a very short period of time. In practice, an example of an impulse would be a hammer striking a surface. Mathematically, a unit impulse is referred to as a Dirac delta function, denoted by $\delta(t)$. It is called a unit impulse because its area is 1. As shown in Fig. 2, the force is applied over the time from 0 to $t_1$. Therefore, as $t_1$ approaches zero, in order for the area to remain equal to 1 the height must approach infinity.

![Dirac delta function](image)

Fig. 2. Dirac delta function.

A general non-unit impulse function can be represented as $A\delta(t)$, where $A$ is its area.

EQUATIONS DESCRIBING SYSTEM RESPONSE

The equation of motion describing the behavior of a second-order mass-spring-dashpot system with a unit impulse input is
\[ \ddot{x} + 2\zeta \omega_n \dot{x} + \omega_n^2 x = \delta(t). \] (1)

The form of the system response will depend on whether the system is under-damped, critically damped, or over-damped. The most straightforward way to solve this differential equation and determine the system response is to use the Laplace transform. The Laplace transform of a Dirac delta function is
\[ L\{\delta(t)\} = 1. \] (2)

**Under-Damped**

For an under-damped system \((\zeta < 1)\), assuming zero initial conditions, the form of the response is
\[ x(t) = \frac{1}{\omega_d} e^{-\zeta \omega_d t} \sin \omega_d t. \] (3)

**Critically Damped**

For a critically damped system \((\zeta = 1)\), and again assuming zero initial conditions, the response is given by
\[ x(t) = t e^{-\omega_d t}. \] (4)

**Over-Damped**

For an over-damped system \((\zeta > 1)\), with zero initial conditions, the response is
\[ x(t) = \frac{1}{2 \omega_n \sqrt{\zeta^2 - 1}} \left[ e^{-\omega_n (\sqrt{\zeta^2 - 1}) t} - e^{-\omega_n (\sqrt{\zeta^2 - 1}) t} \right]. \] (5)

**FORM OF SYSTEM RESPONSE**

The response of a system to an impulse looks identical to its response to an initial velocity. The impulse acts over such a short period of time that it essentially serves to give the system an initial velocity.

Fig. 3 shows the impulse response of three systems: under-damped, critically damped, and over-damped.
Fig. 3. Impulse response of under-damped, critically damped, and over-damped systems.

Table 1 lists the damping ratios of the three systems whose response is shown in Fig. 3.

Table 1. Damping ratios for three example systems.

<table>
<thead>
<tr>
<th>System type</th>
<th>Damping ratio (ζ)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Under-damped</td>
<td>0.22</td>
</tr>
<tr>
<td>Critically damped</td>
<td>1.0</td>
</tr>
<tr>
<td>Over-damped</td>
<td>2.2</td>
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</tbody>
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