Q factor

Nick Lepeshkin

1 Damped oscillations

The equation of motion for a damped oscillator

$$\frac{d^2x}{dt^2} = -\omega_0^2 x - \frac{b}{m}\frac{dx}{dt}$$

leads to the following solution $x(t)$

$$x(t) = A_0 e^{-\frac{t}{2\tau}} cos(\omega t + \phi_0)$$

where A_0 is the initial amplitude, $\tau \equiv m/b$ is the decay time, and frequency $\omega = \sqrt{\omega_0^2 - \left(\frac{1}{2\tau}\right)^2}$. The quality factor Q defined as

$$Q \equiv 2\pi\frac{\tau}{T} = \omega\tau$$

is a convenient measure of how much longer the decay time is compared to the period. Lightly damped oscillations are referred to as high Q, and heavier damped oscillations - as low Q. Informally, the quality factor represents the number of cycles completed by the oscillator before it "rings down" or "runs out of energy". More rigorously, for $Q \gg 1$

$$Q = \pi\frac{A}{\Delta A}$$

where ΔA is the change in amplitude over one period (cycle), or, in terms of energy

$$Q = 2\pi\frac{E}{\Delta E}$$

where ΔE is the decrease in mechanical energy over one period (cycle). Try and convince yourself that all of the expression for Q given above are consistent with each other. Keep in mind, that $e^x \approx 1 + x$ for $x \ll 1$. The frequency of the damped oscillation could also be recast in terms of Q as follows

$$\omega \approx \omega_0\left(1 - \frac{1}{8Q^2}\right)$$

This explains why the variation in frequency due to damping is negligible in most high- and moderate-Q systems.

2 Driven oscillations

Let us analyze the effect of a periodic driving force, $F(t) = F_0 cos(\omega t)$, on a damped oscillator. When the oscillator reaches steady state, the amplitude A is given by

$$A = \frac{F_0/m}{\sqrt{(\omega^2 - \omega_0^2)^2 + (b/m)^2\omega^2}}$$

After some algebraic manipulations and expressing b/m in terms of the Q factor,

$$A = \frac{F_0/m\omega_0^2}{\sqrt{(\omega^2/\omega_0^2 - 1)^2 + \frac{1}{Q^2}\cdot\omega^4/\omega_0^4}}$$

To make this expression easier to interpret, let us look at the expression in the numerator.

$$\Delta L \equiv \frac{F_0}{m\omega_0^2} = \frac{F_0}{k}$$

This is how much the spring is stretched or compressed when a steady force of magnitude F_0 is applied to it. ΔL could also be thought of as the amplitude A at a low (approaching zero) frequency of the driving force. Next, let us define a new variable x parameterizing the frequency so that

$$x \equiv \omega/\omega_0$$

Now,

$$A(x) = \frac{\Delta L}{\sqrt{(x^2-1)^2 + \frac{x^4}{Q^2}}}$$

This function has a maximum at a certain value of x. Setting the derivative of $A(x)$ to zero and solving for x_{max} yields, for $Q \gg 1$,

$$x_{max} = \frac{1}{\sqrt{1+1/Q^2}} \approx 1 - 1/2Q^2 \approx 1$$

or, alternatively,

$$\omega_{max} = \omega_0$$

also known as the resonance condition. What is the amplitude at that frequency?

$$A(x=1) = Q \cdot \Delta L$$

meaning that the amplitude at the resonance frequency is Q times greater than the amplitude off resonance (at zero frequency).

How broad is the resonance, frequency-wise? In other words, what is the bandwidth of the resonance peak? Let us look at the width of the peak at the amplitude equal to $A_{max}/\sqrt{2}$. The energy of the oscillation is proportional to the square of its amplitude, so we are looking for the full width at the half max level (FWHM) of energy. A few more algebraic steps yield the FWHM, Δx to be equal to $1/Q$ which translates into the bandwidth

$$\Delta\omega = \omega_0/Q$$

Thus, Q factor could also be defined as the ratio of the central frequency ω_0 to the bandwidth $\Delta\omega$.

3 Summary

The main point of this write-up is to show how versatile the Q factor is in describing damped oscillations and resonance.

- Q is a measure of damping. It shows how many periods fit in one decay-time interval, or how many periods it takes for the oscillation to ring down, run out of energy.

- Q is a measure of how sharp the resonance is. The resonant amplitude is proportional to Q, whereas the width is inversly proportional to Q.

- Q is also a connecting link between the simple oscillator model and the model of a resonator, a system with a spectrum of natural frequencies which could be selectively excited by applying a periodic external disturbance.