Factoring
Recall that factoring an expression means writing it as a product.

1) Common factors

2) Special formulas

3) Grouping
Common Factors

\[ 3x + 12 = 3 \cdot x + 3 \cdot 4 = 3(x + 4) \]

\[ xy - xy^3 + x^3 y^2 = xy \cdot 1 - xy \cdot y^2 + xy \cdot x^2 y \]
\[ = xy(1 - y^2 + x^2 y) \]

\[ 4x^3 y^4 - 16xy^3 + 20x^3 y^2. \]

Notice that all three terms share a factor of 4, as well as \( x \) and \( y^2 \). So, taking out this common factor of \( 4xy^2 \), we get

\[ 4x^3 y^4 - 16xy^3 + 20x^3 y^2 = 4xy^2(x^2 y^2 - 4y + 5x^2). \]
**Special Formulas**

(i) $x^2 - y^2 = (x + y) \cdot (x - y)$

(ii) $x^3 + y^3 = (x + y) \cdot (x^2 - xy + y^2)$

(iii) $x^3 - y^3 = (x - y) \cdot (x^2 + xy + y^2)$

(iv) $x^2 + (a + b)x + ab = (x + a) \cdot (x + b)$

(v) $acx^2 + (bc + ad)x + bd = (ax + b) \cdot (cx + d)$

(vi) $x^2 + 2xy + y^2 = (x + y)^2$ and $x^2 - 2xy + y^2 = (x - y)^2$
Example: Using Difference of Squares

a) \( s^2 - 16 = (s + 4) \cdot (s - 4) \)

   (Here we are using (i) with \( x = s \) and \( y = 4 \).)

b) \( 9x^2 - 4y^2 = (3x)^2 - (2y)^2 = (3x + 2y) \cdot (3x - 2y) \)

c) \( 5s^2 - 3t^2 = (\sqrt{5s})^2 - (\sqrt{3t})^2 = (\sqrt{5s} + \sqrt{3t}) \cdot (\sqrt{5s} - \sqrt{3t}) \)

d) \( x^4 - y^4 = (x^2 + y^2) \cdot (x^2 - y^2) = (x^2 + y^2)(x + y)(x - y) \)

e) \( 16s^4 - 81t^4 = (4s^2 + 9t^2) \cdot (4s^2 - 9t^2) = (4s^2 + 9t^2)(2s + 3t)(2s - 3t) \)
Example: Using \(x^3+y^3=(x+y) \cdot (x^2-xy+y^2)\)

a) \(27a^3 + 8b^3 = (3a)^3 + (2b)^3 = (3a + 2b) \cdot (9a^2 - 6ab + 4b^2)\)

b) \(64s^3 t^6 + 8y^3 z^6 = (4st^2)^3 + (2yz^2)^3\)
\[= (4st^2 + 2yz^2) \cdot (16s^2 t^4 - 8sy t^2 z^2 + 4y^2 z^4)\]

c) \(a^3 + 2b^3\) is the sum of the cubes of \(a\) and \(\sqrt[3]{2b}\) (which equals \(2^{\frac{1}{3}} b\)).
So,
\[a^3 + 2b^3 = (a + 2^{\frac{1}{3}} b) \cdot (a^2 - 2^{\frac{1}{3}} ab + 2^{\frac{2}{3}} b^2)\).
Using $x^3 - y^3 = (x - y) \cdot (x^2 + xy + y^2)$

Example:

a) $x^3 - 27y^6 = (x^3 - (3y^2)^3) = (x - 3y^2) \cdot (x^2 + 3xy^2 + 9y^4)$

b) $2s^9 - 3t^6 = ((\sqrt[3]{2}s^3)^3 - (\sqrt[3]{3}t^2)^3)$
   
   $= (\sqrt[3]{2}s^3 - \sqrt[3]{3}t^2) \cdot (\sqrt[3]{4}s^6 + \sqrt[3]{6}s^2t^2 + \sqrt[3]{9}t^4)$

Using $x^2 + (a + b)x + ab = (x + a) \cdot (x + b)$

Example: Factor $x^2 + 6x + 8$.

We are looking for two numbers $a$ and $b$ so that their product is 8 and their sum is 6. Well, 8 factors only as 2 times 4, 8 times 1, −2 times −4, or −8 times −1. Of all these possibilities the only pair to add up to 6 is 2 and 4.

So: $x^2 + 6x + 8 = (x + 2)(x + 4)$. 

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Example: Factor $y^2 - 22y + 21$.

First factor 21, so we get the pairs of numbers: 3 and 7, –3 and –7, 21 and 1, and –21 and –1. Picking the two that add up to –22, we use –21 and –1.

$$y^2 - 21y + 21 = (y - 21) \cdot (y - 1)$$

Example: Factor $s^2 + 2s - 15$.

Again factor –15 and find the pair adding to 2. The factors of –15 are –15 and 1, –5 and 3, 5 and –3, or 15 and –1. The correct pair is 2 is 5 and –3, so

$$s^2 + 2s - 15 = (s + 5) \cdot (s - 3).$$
Example: Factor $a^6 - a^3 - 20$.

This looks different, but if we let $x = a^3$ we get $x^2 - x - 20$, and then this method of factoring applies. Now, the factors of $-20$ are of $-20$ and $1$, $-10$ and $2$, $-5$ and $4$, $-4$ and $5$, $-2$ and $10$ $-1$ and $20$. Since their sum has to be $-1$, the only possible choice is $-5$ and $4$. So,

$$x^2 - x - 20 = (x - 5) \cdot (x + 4)$$

and hence

$$a^6 - a^3 - 20 = (a^3 - 5) \cdot (a^3 + 4).$$

This can be factored further to give

$$a^6 - a^3 - 20 = (a^3 - 5) \cdot (a^3 + 4)$$

$$= (a - \sqrt[3]{5})(a^2 + \sqrt[3]{5}a + \sqrt[3]{25})(a + \sqrt[3]{4})(a^2 - \sqrt[3]{4}a + \sqrt[3]{16}) \cdot$$
Grouping

This method sometimes, but not always, does the job. Here’s how it works.

Example: Factor $15xy + 12y + 10x + 8$.

As written, there are no common factors, but notice that the first and second terms have a common factor of $3y$, while the third and fourth terms have a common factor of 2, giving

$$15xy + 12y + 10x + 8 = 3y(5x + 4) + 2(5x + 4).$$

Now these two terms have a common factor of $5x + 4$, which we factor out to get

$$15xy + 12y + 10x + 8 = (5x + 4) \cdot (3y + 2).$$

Notice that the expression is in factored form.
Example: Factor $6ax + 3ay - 4bx - 2by + 10x + 5y$.

You can group in more than one way. Let's try

$$(6ax + 3ay) + (-4bx - 2by) + (10x + 5y)$$

$$= 3a(2x + y) + (-2b)(2x + y) + 5(2x + y)$$

$$= (2x + y) \cdot (3a - 2b + 5).$$
Factor as much as possible.

1) \(2xy + 4x\)  
2) \(6wz + 2wzt\)  
3) \(8xy + 4x + 2w\)  

4) \(6x^2y + 3xy + 9xy^2\)  
5) \(10x^8y^6 + 25x^2y^4 + 20x^3y^{10}\)  
6) \(24x^2yz + 2xy^2z^2 + 4xyz^3\)  

7) \(x^2 - 9\)  
8) \(4y^2 - 9z^2\)  
9) \(16x^4 - y^4\)  
10) \(8s^3 + 27t^3\)  

11) \(8s^3 - 27t^3\)  
12) \(x^2 + 7x + 10\)  
13) \(x^2 + 6x + 8\)  
14) \(x^2 + 6x - 7\)  

15) \(x^2 - 2x - 24\)  
16) \(a^4 - 2a^2 - 24\)  
17) \(s^6 - 7s^3 - 8\)  
18) \(3x^2 + x - 2\)  

19) \(a^2 + 6a + 9\)  
20) \(-2x^2 + 8x - 8\)  
21) \(x^2y^2z + 2xy^2z + y^2z\)  

22) \(3ax + 2ay + 3bx + 2by\)  
23) \(x^4 - x^3y + x - y\)  

24) \(x^{10} + x^6y^2 + x^4y^3 + y^5\)  
25) \(6x^3y - 4xy^3 + 12yx^2 - 8y^3\)  

26) \(3x^2 + 5xy + 7x + 3xy + 5y^2 + 7y\)

Answers
1) \(2x(y + 2)\)  
2) \(2wz(3 + t)\)  
3) \(2x(4y + 2 + w)\)  
4) \(3xy(2x + 1 + 3y)\)  
5) \(5x^2y^4(2x^6y^2 + 5 + 4xy^6)\)  
6) \(2xyz(12x + yz + 2z^2)\)  
7) \((x + 3)(x - 3)\)  
8) \((2y + 3z)(2y - 3z)\)  
9) \((4x^2 + y^2)(4x^2 - y^2) = (4x^2 + y^2)(2x + y)(2x - y)\)  
10) \((2s + 3t)(4s^2 - 6st + 9t^2)\)  
11) \((2s - 3t)(4s^2 + 6st + 9t^2)\)  
12) \((x + 5)(x + 2)\)  
13) \((x + 4)(x + 2)\)  
14) \((x + 7)(x - 1)\)  
15) \((x - 6)(x + 4)\)  
16) \((a^2 - 6)(a^2 + 4) = (a - \sqrt{6})(a + \sqrt{6})(a^2 + 4)\)  
17) \((s^3 - 8)(s^3 + 1) = (s - 2)(s^2 + 2s + 4)(s + 1)(s^2 - s + 1)\)  
18) \((3x - 2)(x + 1)\)  
19) \((a + 3)^2\)  
20) \(-2(x - 2)^2\)  
21) \(y^2z(x + 1)^2\)  
22) \((a + b)(3x + 2y)\)  
23) \((x^3 + 1)(x - y) = (x + 1)(x^2 - x + 1)(x - y)\)  
24) \((x^6 + y^3)(x^4 + y^2) = (x^2 + y^2)(x^4 - x^2y + y^2)(x^4 + y^2)\)  
25) \(2y(x + 2)(3x^2 - 2y^2) = 2y(x + 2)(\sqrt{3}x + \sqrt{2}y)(\sqrt{3}x - \sqrt{2}y)\)  
26) \((x + y)(3x + 5y + 7)\)