Derivatives of Trigonometric Functions

Let $f(x) = \sin x$, then

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{\sin(x + h) - \sin x}{h}$$

Using $\sin(x + h) = \sin x \cos h + \sin h \cos x$

$$f'(x) = \lim_{h \to 0} \frac{\sin x \cos h + \sin h \cos x - \sin x}{h}$$

$$f'(x) = \lim_{h \to 0} \left( \frac{\sin x \cos h - \sin x}{h} \right) + \lim_{h \to 0} \left( \frac{\sin h \cos x}{h} \right)$$

$$f'(x) = \lim_{h \to 0} \left( \sin x \cdot \frac{\cos h - 1}{h} \right) + \lim_{h \to 0} \left( \cos x \cdot \frac{\sin h}{h} \right)$$

Since $\sin x$, and $\cos x$ are constant as far as $h$ is concerned they can be taken out of the limits.
So,

\[ f'(x) = \sin x \cdot \left( \lim_{h \to 0} \frac{\cos h - 1}{h} \right) + \cos x \cdot \left( \lim_{h \to 0} \frac{\sin h}{h} \right) \]

Recall we showed:

\[ \lim_{h \to 0} \frac{\cos h - 1}{h} = 0 \quad \text{and} \quad \lim_{h \to 0} \frac{\sin h}{h} = 1 \]

in which case,

\[ f'(x) = \sin x \cdot \left( \lim_{h \to 0} \frac{\cos h - 1}{h} \right) + \cos x \cdot \left( \lim_{h \to 0} \frac{\sin h}{h} \right) = \sin x \cdot (0) + \cos x \cdot (1) = \cos x \]

So that \((\sin x)' = \cos x\).
For \( g(x) = \cos x \),

\[
g'(x) = \lim_{h \to 0} \frac{g(x + h) - g(x)}{h} = \lim_{h \to 0} \frac{\cos(x + h) - \cos x}{h}
\]

Using \( \cos(x + h) = \cos x \cos h - \sin h \sin x \),

\[
g'(x) = \lim_{h \to 0} \frac{\cos(x + h) - \cos x}{h} = \lim_{h \to 0} \frac{\cos x \cos h - \sin h \sin x - \cos x}{h}
\]

\[
g'(x) = \lim_{h \to 0} \frac{\cos x \cos h - \cos x}{h} - \lim_{h \to 0} \frac{\sin h \sin x}{h}
\]

\[
g'(x) = \cos x \cdot \left( \lim_{h \to 0} \frac{\cos h - 1}{h} \right) - \sin x \cdot \left( \lim_{h \to 0} \frac{\sin h}{h} \right)
\]

\[
g'(x) = \cos x \cdot (0) - \sin x \cdot (1) = -\sin x
\]
To Summarize

\[(\sin x)' = \cos x\]

\[(\cos x)' = -\sin x\]

Examples:

a) If \( f(x) = \sin x + \cos x \), then \( f'(x) = \cos x - \sin x \).

b) If \( f(x) = 2\sin x + 4\cos x \), then \( f'(x) = 2\cos x - 4\sin x \).
Examples:

a) \( f(x) = x^5 \sin x \) then \( f'(x) = 5x^4 \sin x + x^5 \cos x. \)

b) \( f(x) = \cos x \sin x \) then
\[
 f'(x) = (-\sin x) \sin x + \cos x \cos x = \cos^2 x - \sin^2 x = \cos(2x)
\]

c) \( f(x) = x^3 \sin x + x^3 \cos x, \) then
\[
 f''(x) = 3x^2 \sin x + x^3 \cos x + 3x^2 \cos x - x^3 \sin x \\
= 3x^2 (\sin x + \cos x) + x^3 (\cos x - \sin x)
\]

d) \( f(x) = e^x \sin x + e^{-x} \cos x, \) then
\[
 f'(x) = e^x \sin x + e^x \cos x - e^{-x} \cos x - e^{-x} \sin x \\
= (e^x - e^{-x})(\sin x + \cos x)
\]
e) \( f(x) = \frac{x^2 \sin x}{\sin x + \cos x}, \) then
\[
f'(x) = \frac{(2x \sin x + x^2 \cos x)(\sin x + \cos x) - (x^2 \sin x)(\cos x - \sin x)}{(\cos x + \sin x)^2}
\]
\[
= \frac{2x \sin x(\sin x + \cos x) + x^2}{(\cos x + \sin x)^2}
\]
f) \( f(x) = \frac{e^x \sin x}{\cos x}, \) then
\[
f(x) = \frac{e^x \sin x}{\cos x} = \frac{e^x (\sin x + \cos x) \cos x - e^x \sin x (-\sin x)}{\cos^2 x}
\]
\[
= \frac{e^x (\sin x \cos x + 1)}{\cos^2 x}
\]
Derivatives of the Remaining Trigonometric Functions

Since \( \tan x = \frac{\sin x}{\cos x} \), \( \cot x = \frac{\cos x}{\sin x} \), \( \sec x = \frac{1}{\cos x} \), and \( \csc x = \frac{1}{\sin x} \), we can use the quotient rule to compute the derivatives of these functions.

\[
(tan x)' = \frac{\cos x(\cos x) - \sin x(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x
\]

\[
(cot x)' = \frac{\sin x(-\sin x) - \cos x(\cos x)}{\sin^2 x} = -\frac{(\cos^2 x + \sin^2 x)}{\sin^2 x} = -\frac{1}{\sin^2 x} = -\csc^2 x
\]

\[
(sec x)' = \frac{(0)(\cos x) - (1)(-\sin x)}{\cos^2 x} = \frac{\sin x}{\cos^2 x} = \left( \frac{1}{\cos x} \right) \left( \frac{\sin x}{\cos x} \right) = \sec x \tan x
\]

\[
(csc x)' = \frac{(0)(\sin x) - (1)(\cos x)}{\sin^2 x} = -\frac{\cos x}{\sin^2 x} = -\left( \frac{1}{\sin x} \right) \left( \frac{\cos x}{\sin x} \right) = -\csc x \cot x
\]
To summarize:

\[
\begin{align*}
    (\sin x)' &= \cos x \\
    (\cos x)' &= -\sin x \\
    (\tan x)' &= \sec^2 x \\
    (\cot x)' &= -\csc^2 x \\
    (\sec x)' &= \sec x \tan x \\
    (\csc x)' &= -\csc x \cot x
\end{align*}
\]

Examples

a) If \( f(x) = \frac{x}{\sec x} \), then simplify as \( f(x) = x \cos x \), so \( f(x) = \cos x - x \sin x \).

b) Find the equation of the line tangent to the graph of \( f(x) = x \sin x \) at \( x = \pi \).

Since \( f(\pi) = \pi \sin \pi = 0 \) the point is \((\pi,0)\). With 
\[
    f'(x) = \sin x + x \cos x, \quad f'(\pi) = \sin \pi + \pi \cos \pi = -\pi
\]
and so \( m = -\pi \).
The line is \( y = -\pi(x - \pi) \).
c) If \( f(x) = \frac{x^3 + 1}{\tan x + \sec x} \),

\[
 f'(x) = \frac{(3x^2)(\tan x + \sec x) - (x^3 + 1)(\sec^2 x + \sec x \tan x)}{(\tan x + \sec x)^2} = \frac{3x^2 - [x^3 + 1]\sec x}{(\tan x + \sec x)}
\]

d) Find the equation of the line tangent to the graph of \( f(x) = e^x \cos x \) at \( x = 0 \).

Since \( f(0) = 1 \) the point is (0,1). With \( f'(x) = e^x (\cos x - \sin x) \),

\[
 f'(0) = e^0 (\cos 0 - \sin 0) = 1 \text{ and so } m = 1. \text{ The line is } y = x + 1.
\]
e) If \( f(x) = \frac{\cos x + \sec x}{\sin x + \csc x} \)

\[
f'(x) = \frac{(-\sin x + \sec x \tan x)(\sin x + \csc x) - (\cos x + \sec x)(\cos x - \csc x \cot x)}{(\sin x + \csc x)^2}
\]

\[
= \frac{2(\tan^2 x + \cot^2 x) - 1}{(\sin x + \csc x)^2}
\]