Composition:

Consider the function $f(x) = \sqrt{x+1}$. Then

$$f(0) = 1,$$
$$f(1) = \sqrt{2},$$
$$f(x^4) = \sqrt{x^4 + 1},$$
$$f(\sin x) = \sqrt{\sin x + 1},$$
$$f(g(x)) = \sqrt{g(x) + 1},$$

and

$$f(\text{whatever}) = \sqrt{\text{whatever} + 1}.$$
Example: Let \( f(x) = x^3 + 2x \) and \( g(x) = x^4 \). Then what is \( f(g(x)) \)?

\[
f(g(x)) = (g(x))^3 + 2g(x) = (x^4)^3 + 2x^4 = x^{12} + 2x^4
\]

Example: If \( f(x) = \sin x \) and \( g(x) = \sqrt{x + 1} \), then what is \( f(g(x)) \) and \( g(f(x)) \)?

a) \( f(g(x)) = \sin(g(x)) = \sin \sqrt{x + 1} \).

b) \( g(f(x)) = \sqrt{f(x) + 1} = \sqrt{\sin x + 1} \).

Notice that \( f(g(x)) \neq g(f(x)) \).

**Definition:** \( f(g(x)) \) is called the composition of \( f \) with \( g \). Its symbol is \( f \circ g \), and its evaluation at a point \( x \) is denoted \( (f \circ g)(x) \).

Example: Let \( f(x) = \frac{1}{x^2} \), and \( g(x) = \sec x \). Find \( (f \circ g)(x) \).

Solution: \( (f \circ g)(x) = f(g(x)) = \frac{1}{(g(x))^2} = \frac{1}{\sec^2 x} = \cos^2 x \).
Decomposition

In the composition, \( f(g(x)) \), \( f \) is called the outer function, and \( g \) is called the inner function. Taking derivatives of functions that are composites requires using the chain rule. It is important to let \( f \) be the outermost function (because there may be several ways of decomposing). How can you find the outermost function? Answer: if you were to evaluate the function at some point, the last operation you would do corresponds to the outermost function.

Example: Decompose the function \( y(x) = (x^3 + 1)^4 \)

If you were to evaluate this function (at \( x = 1 \) for example), the last operation would be to take the fourth power. So \( f(x) = x^4 \) is the outermost function. Clearly \( g(x) = x^3 + 1 \) is the inner function, and \((f \circ g)(x) = (g(x))^4 = (x^3 + 1)^4 \).
Hence, \( f(x) = x^4, \ g(x) = x^3 + 1 \) gives the desired decomposition.

Example: Decompose the function \( y(x) = \cot^2 x \).

Since \( \cot^2 x \) means \((\cot x)^2\), the last operation is to take the square. So \( f(x) = x^2 \) is the outer function and \( g(x) = \cot x \) is the inner function.

Example: Decompose the function \( y(x) = \sin \sqrt{x} \).

Here you would take \( x \), take its root, and then take sine of the result. So \( f(x) = \sin x \) is the outer function, and \( g(x) = \sqrt{x} \) is the inner function.

Example: Decompose the function \( y(x) = \cos \sqrt{x^6 + 1} \).

The outermost function is \( f(x) = \cos x \), and the inner function is \( g(x) = \sqrt{x^6 + 1} \). (By the way, for the next step in using the chain rule, the function \( g(x) = \sqrt{x^6 + 1} \) must itself be decomposed, which yields the outer function \( \sqrt{x} \) and the inner function \( x^6 + 1 \).)
The Chain Rule

Consider a spherical balloon that is being filled with air. The volume of the balloon is a function of radius, \( V(r) = \frac{4}{3} \pi r^3 \), and the radius is a function of time, \( r = r(t) \). So the volume really is a function of time. This is a typical composition. Given any time \( t \), plugging \( t \) into \( r \) gives the radius and then plugging \( r \) into \( V \) gives the volume. A natural question is how quickly does the balloon fill w.r.t. time.

\[
\begin{align*}
  t & \quad \rightarrow \quad r(t) \\
  & \quad \quad \quad \rightarrow \quad V(r) \\
  & \quad \quad \quad \quad \rightarrow \quad V(r(t))
\end{align*}
\]
Derivation of the Chain Rule

Suppose \( y(x) = (f \circ g)(x) = f[g(x)] \) is a composition function.

\[
y'(x) = \lim_{h \to 0} \frac{y(x + h) - y(x)}{h} = \lim_{h \to 0} \frac{f[g(x + h)] - f[g(x)]}{h}
\]

\[
= \lim_{h \to 0} \frac{f[g(x + h)] - f[g(x)]}{g(x + h) - g(x)} \cdot \frac{g(x + h) - g(x)}{h}
\]

If we write \( g(x + h) = g + \Delta g \), then \( \lim_{h \to 0} g(x + h) - g(x) = \lim_{h \to 0} \Delta g = 0 \), and we can rewrite this last line as

\[
y'(x) = \lim_{\Delta g \to 0} \frac{f[g + \Delta g] - f[g]}{\Delta g} \cdot \lim_{h \to 0} \frac{g(x + h) - g(x)}{h} = f'(g) \cdot g'(x).
\]
So,

\[(f \circ g)'(x) = f'[g(x)] = f''(g) \cdot g'(x)\]

In Leibniz notation

\[
\frac{df}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}
\]

or, if \(y = y(u)\), and \(u = u(x)\), then

\[
\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}
\]

In terms of inside and outside functions

\[
\frac{dy}{dx} = \text{(the derivative of the outside)} \times \text{(the derivative of the inside.)}
\]
Example: Calculate the derivative of \( y = \sin(x^3) \)

Since \( \frac{d}{du} \sin u = \cos u \), and \( \frac{d}{dx}(x^3) = 3x^2 \),

\[
\frac{dy}{dx} = (\cos u)(3x^2) = 3x^2 \cos(x^3)
\]

Example: Calculate the derivative of \( y = (x^2 + 1)^{10} \)

Since \( \frac{d}{du} u^{10} = 10u^9 \), and \( \frac{d}{dx}(x^2 + 1) = 2x \),

\[
\frac{dy}{dx} = (10u^9)(2x) = \left(10(x^2 + 1)^9\right)(2x) = 20x(x^2 + 1)^9
\]
Examples:

a) \[ \frac{d}{dx} [(x^5 + 1)^{100}] = 100(x^5 + 1)^{99}(5x^4) = 500x^4(x^5 + 1)^{99} \]

b) \[ \frac{d}{dx} (\sin(x^2)) = \cos(x^2)(2x) = 2x\cos(x^2) \]

c) \[ \frac{d}{dx} (\cos(x^2)) = -\sin(x^2)(2x) = -2x\sin(x^2) \]

d) \[ \frac{d}{dx} (\sin(e^x)) = \cos(e^x)(e^x) = e^x\cos(e^x) \]

e) \[ \frac{d}{dx} (e^{x^4}) = e^{x^4}(4x^3) = 4x^3e^{x^4} \]

f) \[ \frac{d}{dx} (e^{\tan x}) = e^{\tan x}(\sec^2 x) = \sec^2 x e^{\tan x} \]
g) \[ \frac{d}{dx}(\tan(x^3)) = \sec^2(x^3)(3x^2) = 3x^2 \sec^2(x^3) \]

h) \[ \frac{d}{dx}(\sec(\sin x)) = \sec(\sin x) \tan(\sin x)(\cos x) = \cos x \sec(\sin x) \tan(\sin x) \]

i) \[ \frac{d}{dx}(\sin(\cos x)) = \cos(\cos x) \cdot (-\sin x) = -\sin x \cos(\cos x) \]

j) \[ \frac{d}{dx}(\sqrt{x^2 + 1}) = \frac{d}{dx}(\sqrt{x^2 + 1}) = \frac{1}{2}(x^2 + 1)^{-\frac{1}{2}}(2x) = \frac{x}{\sqrt{x^2 + 1}} \]
Sometimes you will have to apply the chain rule more than once.

Examples:

a) 
\[
\frac{d}{dx} \left( e^{\sin x^4} \right) = e^{\sin(x^4)} \frac{d}{dx} (\sin x^4) = e^{\sin(x^4)} (\cos x^4) \frac{d}{dx} (x^4). \\
= e^{\sin(x^4)} (\cos x^4) (4 x^3) = 4 x^3 \cos x^4 \ e^{\sin(x^4)}
\]

b) 
\[
\frac{d}{dx} \left( \sin(e^{3x^2+x}) \right) = \cos(e^{3x^2+x}) \frac{d}{dx} \left( e^{3x^2+x} \right) \\
= \cos(e^{3x^2+x}) e^{3x^2+x} \frac{d}{dx} (3x^2 + x). \\
= \cos(e^{3x^2+x}) e^{3x^2+x} (6x + 1) \cdot = (6x + 1) e^{3x^2+x} \cos(e^{3x^2+x})
\]
c) \[
\frac{d}{dx} (\sin^2(4x)) = \frac{d}{dx} ((\sin(4x))^2) = 2\sin(4x) \cdot \frac{d}{dx} \sin(4x)
\]
\[= 2\sin(4x)\cos(4x) \cdot (4)\]
\[= 8\sin(4x)\cos(4x)\]

d) \[
\frac{d}{dx} (\cos^4 \sqrt{x}) = 4\cos^3 \sqrt{x} \cdot \frac{d}{dx} \cos \sqrt{x} = 4\cos^3 \sqrt{x} \cdot (-\sin \sqrt{x}) \frac{d}{dx} \sqrt{x}
\]
\[= -\frac{2}{\sqrt{x}} \cos^3 \sqrt{x} \cdot \sin \sqrt{x}\]
This “nesting” of functions can get ugly, but the chain rule still works.

Example:

\[
\frac{d}{dx} \left( [1 + \cos \sqrt{x^2 + 1}]^4 \right) = 4[1 + \cos \sqrt{x^2 + 1}]^3 \cdot \frac{d}{dx} \left[ 1 + \cos \sqrt{x^2 + 1} \right]
\]

\[
= 4[1 + \cos \sqrt{x^2 + 1}]^3 \cdot \left[ \frac{d}{dx} \cos \sqrt{x^2 + 1} \right]
\]

\[
= 4[1 + \cos \sqrt{x^2 + 1}]^3 \cdot \left[ (-\sin \sqrt{x^2 + 1}) \cdot \frac{d}{dx} \sqrt{x^2 + 1} \right]
\]

\[
= 4[1 + \cos \sqrt{x^2 + 1}]^3 \cdot \left[ (-\sin \sqrt{x^2 + 1}) \cdot \frac{1}{2\sqrt{x^2 + 1}} \frac{d}{dx} (x^2 + 1) \right]
\]

\[
= 4[1 + \cos \sqrt{x^2 + 1}]^3 \cdot \left[ (-\sin \sqrt{x^2 + 1}) \cdot \frac{1}{2\sqrt{x^2 + 1}} (2x) \right]
\]

\[
= -\frac{4x \sin \sqrt{x^2 + 1}[1 + \cos \sqrt{x^2 + 1}]^3}{\sqrt{x^2 + 1}}
\]
Generalized Derivatives

\[
\frac{d}{dx} \left[ (u(x))^n \right] = n(u(x))^{n-1} \cdot u'(x) \quad \text{..........} \quad \frac{d}{dx} [e^{u(x)}] = u'(x)e^{u(x)}
\]

\[
\frac{d}{dx} \sin(u(x)) = u'(x)\cos(u(x)) \quad \frac{d}{dx} \cos(u(x)) = -u'(x)\sin(u(x))
\]

\[
\frac{d}{dx} \tan(u(x)) = u'(x)\sec^2(u(x)) \quad \frac{d}{dx} \cot(u(x)) = -u'(x)\csc^2(u(x))
\]

\[
\frac{d}{dx} \sec(u(x)) = u'(x)\sec(u(x))\tan(u(x))
\]

\[
\frac{d}{dx} \csc(u(x)) = -u'(x)\csc(u(x))\cot(u(x))
\]
Examples:
\[ y = \sin(x^2 + 1) \quad y' = 2x \cos(x^2 + 1) \]
\[ y = \sin(x^4 + x) \quad y' = (4x^3 + 1) \cos(x^4 + x) \]
\[ f(x) = e^{x^2} \quad f'(x) = 2xe^{x^2} \]
\[ f(x) = e^{\sin x + \cos x} \quad f'(x) = (\cos x - \sin x)e^{\sin x + \cos x} \]
\[ y = \cot(x^5 + 5x) \quad y' = -(5x^4 + 5)\csc^2(x^5 + 5x) \]
In particular

\[
\frac{d}{dx} \sin(\omega x) = \omega \cos(\omega x) \quad \frac{d}{dx} \cos(\omega x) = -\omega \sin(\omega x) \quad \frac{d}{dx} [e^{kx}] = k e^{kx}
\]

Examples

\[\begin{align*}
y &= \sin \pi x & y' &= \pi \cos \pi x \\
y &= \cos 2x & y' &= -2 \sin 2x \\
f(x) &= e^{4x} & f'(x) &= 4e^{4x} \\
f(x) &= e^{-x/2} & f(x) &= -\frac{1}{2}e^{-x/2}
\end{align*}\]