Average and Instantaneous Speed

A moving body’s average speed during a time interval is defined as the distance covered divided by the elapsed time. As this time interval gets smaller and smaller the value we approach is called the instantaneous speed.

Suppose \( y = 16t^2 \) is the distance (in feet) an object has moved in a time interval (in seconds), then computing the change in \( y \) (denoted \( \Delta y \)) and the time elapsed (denoted \( \Delta t \)) one can calculate the average velocity in that time interval.

For example the average velocity in the first 2 seconds is
\[
\frac{\Delta y}{\Delta t} = \frac{16(2^2) - 16(0)}{2 - 0} = 32 \text{ ft/sec}
\]

The average velocity during the 1 second interval between second 1 and second 2 is
\[
\frac{\Delta y}{\Delta t} = \frac{16(2^2) - 16(1)}{2 - 1} = 48 \text{ ft/sec}
\]
By letting the time interval get smaller and smaller, we can define the instantaneous velocity or a velocity at one point. Consider the time interval 
\[ [t_0, t_0 + \Delta t], \]
then the change in distance divided by the change in time is

\[
\frac{\Delta y}{\Delta t} = \frac{16(t_0 + \Delta t)^2 - 16(t_0)^2}{\Delta t}
\]

Let’s compute the average speed at the time \( t_0 = 1 \) for different values of \( \Delta t \).

<table>
<thead>
<tr>
<th>Length of time interval ( \Delta t )</th>
<th>Average Speed at ( t_0 = 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>48</td>
</tr>
<tr>
<td>0.1</td>
<td>33.6</td>
</tr>
<tr>
<td>0.01</td>
<td>32.16</td>
</tr>
<tr>
<td>0.001</td>
<td>32.016</td>
</tr>
<tr>
<td>0.0001</td>
<td>32.0016</td>
</tr>
</tbody>
</table>

The average velocity certainly seems like it gets closer and closer to 32 ft/sec.
Average Rates of Change

Given a function \( y = f(x) \) the average rate of change over the interval \((a, b)\) is

\[
\frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a} = \frac{f(a + \Delta x) - f(a)}{\Delta x}, \quad \Delta x \neq 0
\]

Where we assume \( b = a + \Delta x \) for some value of \( \Delta x \).

Example: Find the average rate of change of the function \( f(x) = x^2 \) over the intervals \((0, 1)\) and \((2, 4)\)?

For \((0, 1)\):

\[
\frac{\Delta y}{\Delta x} = \frac{f(1) - f(0)}{1 - 0} = \frac{1 - 0}{1 - 0} = 1
\]

For \((2, 4)\):

\[
\frac{\Delta y}{\Delta x} = \frac{f(4) - f(2)}{4 - 2} = \frac{16 - 4}{4 - 2} = 6
\]
Slopes of Tangent Lines

Consider the graph of $f(x) = x^2 + x$ shown below.

![Graph of $f(x) = x^2 + x$]
At $x = 1$, the slope seems to be approximately 3. Since $f(1) = 2$, the tangent line has equation $y = 3x - 1$.

Consider the graph of a function $y = f(x)$ shown below. Suppose $P$ and $Q$ are two points on the graph corresponding to $x = a$, and $x = b = a + h$.

$P$ is the point $(a, f(a))$. $Q$ is the point $(a + h, f(a + h))$. The line drawn through these two points is called the **secant line**, and it has slope:

$$m_{sec} = \frac{f(a + h) - f(a)}{(a + h) - a} = \frac{f(a + h) - f(a)}{h}$$
The following table computes the slopes of several secant lines for the function \( f(x) = x^2 + x \) beginning at \( x = 1 \) for different small intervals.

Note: \( \Delta y = f(1+h) - f(1) \)

<table>
<thead>
<tr>
<th>Interval</th>
<th>[1, 1.1]</th>
<th>[1, 1.01]</th>
<th>[1, 1.001]</th>
<th>[1, 1.0001]</th>
<th>[1, 1.00001]</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Delta x = h )</td>
<td>0.1</td>
<td>0.01</td>
<td>0.001</td>
<td>0.0001</td>
<td>0.00001</td>
</tr>
<tr>
<td>( \Delta y )</td>
<td>0.31</td>
<td>0.0301</td>
<td>0.003001</td>
<td>0.0003001</td>
<td>0.000030001</td>
</tr>
<tr>
<td>Slope = ( \frac{\Delta y}{\Delta x} )</td>
<td>3.1</td>
<td>3.01</td>
<td>3.001</td>
<td>3.0001</td>
<td>3.00001</td>
</tr>
</tbody>
</table>

As \( \Delta x = h \) approaches 0, the slopes in the last line certainly appear to approach 3.

To show this consider that the secant line joining the two points \( x = 1 \) and \( x = 1 + h \) has slope

\[
\frac{\Delta y}{\Delta x} = \frac{f(1+h) - f(1)}{(1+h) - 1} = \frac{f(1+h) - f(1)}{h}
\]
Using the function \( f(x) = x^2 + x \)

\[
\frac{f(1+h) - f(1)}{h} = \frac{[(1+h)^2 + (1+h)] - 2}{h} = \frac{3h + h^2}{h} = 3 + h
\]

Since

\[
\frac{\Delta y}{\Delta x} = \frac{f(1+h) - f(1)}{h} = (3 + h).
\]

As \( h \) gets smaller and smaller, \( \frac{\Delta y}{\Delta x} \) gets closer and closer to 3 which is what we expected. Hence the instantaneous rate of change of the function \( f(x) = x^2 + x \) at \( x = 1 \) is defined to be 3.
Example: Using this method find the instantaneous rate of change of the function \( y = f(x) = x^2 - 2x + 1 \) at \( x = -1 \).

The average rate of change of \( f \) over the interval \([-1, -1+h] \) is

\[
\frac{\Delta y}{\Delta x} = \frac{f(-1+h) - f(-1)}{h}
\]

Hence

\[
\frac{\Delta y}{\Delta x} = \left[(-1+h)^2 - 2(-1+h) + 1\right] - 4 = -4h + h^2 = -4 + h
\]

When \( h \) gets smaller and smaller \( \frac{\Delta y}{\Delta x} \) gets closer and closer to \(-4\), hence the instantaneous rate of change is defined to be \(-4\).
Example: Using this method find the instantaneous rate of change of the function \( y = f(x) = \frac{1}{x} \) at \( x = 2 \).

The average rate of change of \( f \) over the interval \([2, 2+h]\) is

\[
\frac{\Delta y}{\Delta x} = \frac{f(2+h)-f(2)}{h}
\]

Hence

\[
\frac{\Delta y}{\Delta x} = \frac{1}{2+h} - \frac{1}{2} = \frac{2-(2+h)}{2(2+h)} = \frac{-h}{2(2+h)} = \frac{-1}{2(2+h)}
\]

When \( h \) gets smaller and smaller \( \frac{\Delta y}{\Delta x} \) gets closer and closer to \(-1/4\), hence the instantaneous rate of change is defined to be \(-1/4\).