1) a) Write down the limit definition of the derivative of a function \( f'(x) \).
    b) Using part (a) calculate \( f'(x) \), for \( f(x) = 6x + 2 \). How does the result compare to what you already know about linear functions?

2) a) Write down the limit definition of the derivative of a function \( f'(x) \).
    b) Using part (a) calculate \( f'(x) \), for \( f(x) = 2x^2 + 4x - 3 \).
    c) Write down the power rule, and use it to check your answer in part (b).

3) a) Write down the limit definition of the derivative of a function \( f'(x) \).
    b) Using part (a) calculate \( f'(x) \), for \( f(x) = 3x^2 - 3x + 2 \).
    c) Write down the power rule, and use it to check your answer in part (b).

4) a) Using the limit definition of the derivative, calculate \( f'(x) \), for \( f(x) = \frac{1}{x} + \frac{x^3}{3} \).
    b) Use the power rule to check your answer.

5) a) Using the limit definition of the derivative, calculate \( f'(x) \), for \( f(x) = \frac{1}{x^2} + \frac{x^2}{2} \).
    b) Use the power rule to check your answer.

6) Consider the function \( f(x) = (x - 2 | x |)^2 + \pi^3 \).
    a) Write down the limit definition for the derivative of a function.
    b) Using the definition of the derivative, determine whether or not the function \( f(x) \) differentiable at \( x = 0 \). **Justify your steps using limit laws or theorems.**
    c) Is \( f(x) \) continuous at \( x = 0 \)? Explain your reasoning.

7) Consider the function \( f(x) = (x + | x |)^2 + 1 \).
    a) Using the definition of the derivative, determine whether or not the function \( f(x) \) differentiable at \( x = 0 \).
    b) Is \( f(x) \) continuous at \( x = 0 \)? Explain your reasoning.

8) Suppose that the function \( f(x) \) is defined by the rule
\[
    f(x) = \begin{cases} 
    ax^2 + 1 & x \leq 2 \\
    x^3 + a & x > 2
    \end{cases}
\]
where \( a \) is a real constant. What must the value of \( a \) be in order for the function to be continuous at \( x = 2 \)? Justify your reasoning.

9) Suppose that the function \( f(x) \) is defined by the rule
\[
    f(x) = \begin{cases} 
    x^2 + a & x < 1 \\
    bx + 1 & x \geq 1
    \end{cases}
\]
What relationship must \( a \) and \( b \) satisfy for the function to be continuous at \( x = 1 \)?

10) Using the limit definition of the derivative, calculate \( f'(x) \), for \( f(x) = x^{\frac{3}{2}} \). Use the power rule to check your answer.
Solutions

1) a) \[ f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \]

b) \[ f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{6(x+h) + 2 - 6x - 2}{h} \]
\[ = \lim_{h \to 0} \frac{6x + 6h + 2 - 6x - 2}{h} \]
\[ = \lim_{h \to 0} \frac{6h}{h} = 6 \]

Since \( f(x) = 6x + 2 \) is a linear function its graph is a line whose constant slope is 6. (Slope = Derivative)

2) a) \[ f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \]

b) \[ f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{2(x+h)^2 + 4(x+h) - 3 - [2x^2 + 4x - 3]}{h} \]
\[ = \lim_{h \to 0} \frac{2x^2 + 4xh + 2h^2 + 4x + 4h - 3 - 2x^2 - 4x + 3}{h} \]
\[ = \lim_{h \to 0} \frac{4xh + 2h^2 + 4h}{h} = \lim_{h \to 0} 4x + 2h + 4 = 4x + 4 \]

c) Power rule is \( \frac{d}{dx} x^k = kx^{k-1} \). With \( f(x) = 2x^2 + 4x - 3 \)
\[ f'(x) = \frac{d}{dx} (2x^2 + 4x - 3) = (2)(2x) + 4(1) - 0 = 4x + 4 \]

3) a) \[ f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \]

b) \[ f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{3(x+h)^2 - 3(x+h) + 2 - [3x^2 - 3x + 2]}{h} \]
\[ = \lim_{h \to 0} \frac{3x^2 + 6xh + 3h^2 - 3x - 3h + 2 - 3x^2 + 3x - 2}{h} \]
\[ = \lim_{h \to 0} \frac{6xh + 3h^2 - 3h}{h} = \lim_{h \to 0} 6x + 3h - 3 = 6x - 3 \]

c) Power rule is \( \frac{d}{dx} x^k = kx^{k-1} \). With \( f(x) = 3x^2 - 3x + 2 \)
\[ f'(x) = \frac{d}{dx} (3x^2 - 3x + 2) = (3)(2x) - 3(1) - 0 = 6x - 3 \]
4) a) \( f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\left(\frac{1}{x+h} + \frac{(x+h)^3}{3}\right) - \left(\frac{1}{x} + \frac{x^3}{3}\right)}{h} \)

\[
= \lim_{h \to 0} \frac{\left[\frac{1}{x+h} - \frac{1}{x}\right] + \frac{(x+h)^3 - x^3}{3}}{h} = \lim_{h \to 0} \frac{\left[\frac{1}{x+h} - \frac{1}{x}\right] + \left[\frac{(x+h)^3 - x^3}{3}\right]}{h} \\
= \lim_{h \to 0} \left[\frac{x - (x+h)}{x(x+h)}\right] + \lim_{h \to 0} \left[\frac{3hx^2 + 3h^2 x + h^3 - x^3}{3h}\right] = -\frac{1}{x^2} + x^2
\]

b) Power rule is \( \frac{d}{dx} x^k = kx^{k-1} \). With \( f(x) = \frac{1}{x} + \frac{x^3}{3} = -\frac{1}{x^2} + x^2 \)

\[
f'(x) = \frac{d}{dx} \left( x^{-1} + \frac{1}{3} x^3 \right) = (-1)(x^{-2}) + \frac{1}{3}(3x^2) = -\frac{1}{x^2} + x^2
\]

5) a) \( f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\left(\frac{1}{(x+h)^2} + \frac{(x+h)^2}{2}\right) - \left(\frac{1}{x^2} + \frac{x^2}{2}\right)}{h} \)

\[
= \lim_{h \to 0} \frac{\left[\frac{1}{(x+h)^2} - \frac{1}{x^2}\right] + \frac{(x+h)^2}{2} - \frac{x^2}{2}}{h} = \lim_{h \to 0} \frac{\left[\frac{1}{(x+h)^2} - \frac{1}{x^2}\right]}{h} + \lim_{h \to 0} \left[\frac{(x+h)^2 - x^2}{2}\right] \\
= \lim_{h \to 0} \left[\frac{x^2 - (x+h)^2}{x^2(x+h)^2}\right] + \lim_{h \to 0} \left[\frac{x^2 + 2xh + h^2}{2} - \frac{x^2}{2}\right] = -\frac{2}{x^3} + x
\]

b) Power rule is \( \frac{d}{dx} x^k = kx^{k-1} \). With \( f(x) = \frac{1}{x^2} + \frac{x^2}{2} = x^{-2} + \frac{1}{2} x^2 \)

\[
f'(x) = \frac{d}{dx} \left( x^{-2} + \frac{1}{2} x^2 \right) = (-2)(x^{-3}) + \frac{1}{2}(2x) = -\frac{2}{x^3} + x
\]
Consider the function \( f(x) = (x - 2| x |)^2 + \pi^3 \).

a) Write down the limit definition for the derivative of a function \( f(x) \) at \( x = 0 \), that is \( f'(0) \).

\[
f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h}
\]

b) Using the definition of the derivative, determine whether or not the function \( f(x) \) differentiable at \( x = 0 \). Justify your steps using limit laws or theorems.

\[
f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{(h - 2|h|)^2}{h}
\]

Since \( |h| \) is piecewise defined as \( |h| = \begin{cases} -h & h < 0 \\ h & h \geq 0 \end{cases} \), one has to examine both the left hand and right hand limits to see if \( f'(0) = \lim_{h \to 0} \frac{(h - 2|h|)^2}{h} \) exists.

\[
\lim_{h \to 0^+} \frac{(h - 2|h|)^2}{h} = \lim_{h \to 0^+} \frac{(h-2)^2}{h} = \lim_{h \to 0^+} \frac{(-h)^2}{h} = \lim_{h \to 0^+} \frac{h^2}{h} = \lim_{h \to 0^+} h = 0 \quad \text{by limit law 8.}
\]

\[
\lim_{h \to 0^-} \frac{(h - 2|h|)^2}{h} = \lim_{h \to 0^-} \frac{(h+2)^2}{h} = \lim_{h \to 0^-} \frac{9h^2}{h} = \lim_{h \to 0^-} \frac{9h^2}{h} = \lim_{h \to 0^-} 9h = 0 \quad \text{by limit laws 4, 7, & 8.}
\]

Since the left and right hand limits are both 0, the limit exists, and \( f'(0) = 0 \). Since the derivative exists at \( x = 0 \), \( f(x) \) differentiable at \( x = 0 \).

c) Is \( f(x) \) continuous at \( x = 0 \)? Justify your answer. Yes since it is differentiable at \( x = 0 \), and differentiability implies continuity.

7) a) Using the limit definition of the derivative, we need to show whether, or not, \( f'(0) \) exists.

\[
f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{[(h+|h|)^2 + 1] - [1]}{h} = \lim_{h \to 0} \frac{(h+|h|)^2}{h}.
\]

The definition of the absolute value of \( x \) is \( |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases} \).

We now consider the left and right-hand limits. If they both exist and are the same value then the limit above exists and is the same value.

\[
\lim_{h \to 0^+} \frac{(h-h)^2}{h} = \lim_{h \to 0^+} \frac{0}{h} = 0 \quad \text{and} \quad \lim_{h \to 0^-} \frac{(h+h)^2}{h} = \lim_{h \to 0^-} \frac{4h^2}{h} = \lim_{h \to 0^-} 4h = 0
\]

Since the left and right hand derivatives are both equal to 0, the function is differentiable at \( x = 0 \), and \( f'(0) = 0 \).

b) Since differentiability at a point implies continuity at that point, and because \( f(x) \) is differentiable at \( x = 0 \), the function \( f(x) \) is continuous at \( x = 0 \).
Some will, no doubt, use the definition of continuity. If so:

The definition of continuity at \( x = 0 \) is \( \lim_{x \to 0} f(x) = f(0) \).

First off the function is defined \( x = 0 \). \( f(0) = (0 + |0|)^2 + 1 = 1 \).

Now, to show the limit exists consider the left and right limits:

\[
\lim_{x \to 0} f(x) = \lim_{x \to 0} [(x - x)^2 + 1] = \lim_{x \to 0} 1 = 1 ,
\]
\[
\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} [(x + x)^2 + 1] = \lim_{x \to 0^+} (4x^2 + 1) = 1.
\]

Since the left-hand and right-hand limit are equal, \( \lim_{x \to 0} f(x) = 1 = f(0) \), so \( f(x) \) is continuous at \( x = 0 \).

Here’s another possibility:
Since \( x \) and \(|x|\) are both continuous at \( x = 0 \), \( g(x) = x + |x| \) is continuous at \( x = 0 \).
Since \( g(x) = x + |x| \) is continuous at \( x = 0 \), and \( x^2 + 1 \) is continuous everywhere (it’s a polynomial,) \( h(x) = f(g(x)) = (x + |x|)^2 + 1 \) is continuous at \( x = 0 \). (Using Continuity of a Composition Theorem.)

8) The definition for \( f(x) \) to be continuous at \( x = 2 \) is: \( \lim_{x \to 2} f(x) = f(2) \).

i) \( f(2) = 4a + 1 \), so \( f \) is defined at 2.

ii) For \( \lim_{x \to 2} f(x) \) to exist, both the left and right hand limits must exist and be equal.

\[
\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} (x^3 + a) = 8 + a ,
\]

since \( x^3 + a \) is a polynomial, and hence continuous.

\[
\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} (ax^2 + 1) = 4a + 1 ,
\]

since \( ax^2 + 1 \) is a polynomial, and hence continuous.

so \( \lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) \) iff \( 8 + a = 4a + 1 \), or \( a = \frac{7}{3} \).

iii) Since \( \lim_{x \to 2} f(x) = \frac{7}{3} = f(2) \), \( f \) is continuous at \( x = 2 \).
9) Suppose that the function \( f(x) \) is defined by the rule
\[
 f(x) = \begin{cases} 
 x^2 + a & \text{if } x < 1 \\
 b x + 1 & \text{if } x \geq 1 
\end{cases}
\]

What relationship must \( a \) and \( b \) satisfy for the function to be continuous at \( x = 1 \)?

The definition of continuity of \( f(x) \) at \( x = 1 \), is \( \lim_{x \to 1} f(x) = f(1) \).

From the rule above \( f(1) = b + 1 \).

As for the limit; For the limit \( \lim_{x \to 1} f(x) \) to exist, both the left and right hand limits must exist and be equal.

The Right Hand Limit:
\[
\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (b x + 1) = b + 1 , \quad \text{since } b x + 1 \text{ is a polynomial, and hence continuous everywhere}
\]

so that the limit is indeed the definition, i.e. you can simply plug \( x = 1 \) into \( b x + 1 \). Alternatively you could use the limit laws.

\[
\lim_{x \to 1^+} (b x + 1) = \lim_{x \to 1^+} (b x) + \lim_{x \to 1^+} (1) = b \lim_{x \to 1^+} (x) + \lim_{x \to 1^+} (1) = b \cdot 1 + 1 = b + 1
\]

The left hand limit
\[
\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^2 + a) = 1 + a , \quad \text{since } x^2 + a \text{ is a polynomial, and hence continuous everywhere}
\]

so that the limit is indeed the definition, i.e. you can simply plug \( x = 1 \) into \( x^2 + a \). In this case too, you could use the limit laws.

\[
\lim_{x \to 1^-} (x^2 + a) = \lim_{x \to 1^-} (x^2) + \lim_{x \to 1^-} (a) = \left( \lim_{x \to 1^-} (x) \right)^2 + \lim_{x \to 1^-} (a) = 1^2 + a = 1 + a
\]

Setting the LH and RH limits equal gives \( b + 1 = 1 + a \) or \( a = b \).

10) Since \( f(x) = x^{\frac{3}{2}} \), \( f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{(x+h)^3} - \sqrt{x^3}}{h} \).

Conjugating this expression we get:
\[
f'(x) = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h(\sqrt{(x+h)^3} + \sqrt{x^3})} = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}\sqrt{(x+h)^3} + \sqrt{x^3}
\]
\[
= \lim_{h \to 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3) - x^3}{h}\sqrt{(x+h)^3} + \sqrt{x^3}
\]
\[
= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h}\sqrt{(x+h)^3} + \sqrt{x^3}
\]
\[
= \lim_{h \to 0} \frac{3x^2 + 3xh + h^2}{\sqrt{(x+h)^3} + \sqrt{x^3}} = \frac{3x^2}{\sqrt{3} + \sqrt{x^3}} = \frac{3x^2}{2} \cdot \frac{1}{x^{\frac{3}{2}}} = \frac{3}{2} \cdot x^{\frac{3}{2}}
\]

Power Rule is \( \frac{d}{dx} (x^k) = kx^{k-1} \), so \( \frac{d}{dx} (x^{\frac{3}{2}}) = \frac{3}{2} (x^{\frac{3}{2}-1}) = \frac{3}{2} x^{\frac{1}{2}} \).