For Questions 1-4: i) On what intervals is the function \( f(x) \) increasing? Decreasing?  ii) What are the relative extrema?  iii) On what intervals is the function \( f(x) \) concave up? Concave down?   iv) What are the points of inflection?  v) What is the equation of the line tangent to the graph of \( f(x) \) at \( x = 1 \)?

1) \( f(x) = \frac{1}{4} x^4 + \frac{1}{2} x^2 \)

2) \( f(x) = \frac{x}{x^2 + 4} \)

3) \( f(x) = x^3 e^{-2x} \)

4) \( f(x) = x^4 \ln x \)

5) Consider the function \( f(x) = x^3 - 3x^2 - 9x \).
   a) Find the intervals of increase or and decrease.
   b) Find the local maximum and minimum values.
   c) Find the intervals of concavity and any points of inflection.

6) Let \( f(x) = x^2 e^x \).
   a) Determine intervals on which \( f \) is increasing and decreasing.
   b) Determine all relative extrema of \( f \).
   c) Determine intervals on which \( f \) is concave up and concave down.
   d) Determine any inflection points of \( f \).

7) Let \( f(x) = x^4 e^{-x} \).
   a) Determine intervals on which \( f \) is increasing and decreasing.
   b) Determine all relative extrema of \( f \).
   c) Determine intervals on which \( f \) is concave up and concave down.
   d) Determine any inflection points of \( f \).

8) Suppose that \( f(x) = \frac{1}{4} x^4 - \frac{3}{2} x^3 + 3x^2 + 1 \)
   a) On what intervals is the function \( f(x) \) increasing? Decreasing?
   b) On what intervals is the function \( f(x) \) concave up? Concave down?
   c) What are the points of inflection? (If, any.)
   d) What is the equation of the line tangent to the graph of \( f(x) \) at \( x = 1 \)?
Solutions:

1) \( f(x) = \frac{1}{4}x^4 + \frac{1}{2}x^2 \)

i) Increasing for \( x > 0 \), and decreasing for \( x < 0 \)  ii) Relative min at \( x = 0 \) of 1.

iii) Concave up everywhere.  iv) Since concavity never changes, there are no points of inflection.

v) Since \( f(l) = \frac{3}{4} \) and \( f'(l) = 2 \), \( y = 2x - \frac{5}{4} \)

2) \( f(x) = \frac{x}{x^2 + 4} \)

i) Increasing on \((-2, 2)\), Decreasing on \((-\infty, -2) \cup (2, \infty)\).

ii) Relative min of \(-\frac{1}{4}\) at \( x = -2 \). Relative max of \( \frac{1}{4} \) at \( x = 2 \).

iii) Concave up on \((-2\sqrt{3}, 0) \cup (2\sqrt{3}, \infty)\), Concave down on \((-\infty, -2\sqrt{3}) \cup (0, 2\sqrt{3})\)

iv) Points of inflection; \((0, 0),\left(-2\sqrt{3}, -\frac{\sqrt{3}}{8}\right),\) and \(\left(2\sqrt{3}, \frac{\sqrt{3}}{8}\right)\)

v) Since \( f'(l) = \frac{3}{25} \), \( y = \frac{3}{25}x + \frac{2}{25} \)

3) \( f(x) = x^3e^{-2x} \)

i) Increasing on \((-\infty, -\frac{3}{2})\), Decreasing on \(\left(-\frac{3}{2}, \infty\right)\). ii) Relative max of \( \frac{27}{8e^3} \) at \( x = -\frac{3}{2} \).

iii) Concave up on \(\left(0, \frac{3-\sqrt{3}}{2}\right) \cup \left(\frac{3+\sqrt{3}}{2}, \infty\right)\), Concave down on \((-\infty, 0) \cup \left(\frac{3-\sqrt{3}}{2}, \frac{3+\sqrt{3}}{2}\right)\).

iv) Points of inflection are at \((0, 0),\left(\frac{3+\sqrt{3}}{2}, \frac{27 + 15\sqrt{3}}{4}e^{-3\sqrt{3}}\right),\) \(\left(\frac{3-\sqrt{3}}{2}, \frac{27 - 15\sqrt{3}}{4}e^{-3\sqrt{3}}\right)\).

v) Since \( f'(l) = e^{-2} \), \( y = e^{-2}x \).

4) \( f(x) = x^4 \ln x \)

i) Increasing on \(\left(e^{-1/4}, \infty\right)\), Decreasing on \(\left(0, e^{-1/4}\right)\). ii) Relative min at \( x = e^{-1/4} \) of \( -\frac{1}{4e} \).

iii) Concave up on \(\left(e^{-7/12}, \infty\right)\), Concave down on \(\left(0, e^{-7/12}\right)\)

iv) Points of inflection \(\left(e^{-7/12}, -\frac{7}{12e^{7/3}}\right)\)

v) Since \( f'(l) = 1 \), \( y = x - 1 \).
5) a) \[ f'(x) = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 3(x - 3)(x + 1) \].
\[ f'(x) > 0 \text{ for } (-\infty, -1) \cup (3, \infty) \text{ and so } f(x) \text{ increases on } (-\infty, -1) \cup (3, \infty). \]
\[ f'(x) < 0 \text{ for } (-1, 3) \text{ and so } f(x) \text{ decreases on } (-1, 3). \]

b) Critical numbers are \( x = -1 \) and \( x = 3 \). \( f(-1) = 5 \) is local maximum value. \( f(3) = -27 \) is local minimum value.

c) \( f''(x) = 6x - 6 = 6(x - 1) \) \( f(x) \) is concave up on \( x > 1 \). \( f(x) \) is concave down on \( x < 1 \). Since the concavity changes at \( x = 1 \), \((1, -11)\) is the only point of inflection.

6) a) \[ f'(x) = 2xe^{-x} + x^2e^{-x} = (2x + x^2)e^{-x} = x(2 + x)e^{-x} \] So \( f'(x) = 0 \) at \( x = 0 \), and \( x = -2 \)

\[ f'(x) > 0 \text{ for } x < -2 \text{ and } x > 0 \text{, and so } f \text{ is increasing on those intervals.} \]

\[ f'(x) < 0 \text{ for } -2 < x < 0 \text{, and so } f \text{ is decreasing on that interval.} \]

b) At \( x = 0 \), since \( f'(x) < 0 \) for \( x < 0 \), and \( f'(x) > 0 \) for \( x > 0 \), the first derivative test says that \( f(0) = 0 \) is a local minimum value. At \( x = -2 \), \( f'(x) > 0 \) for \( x < -2 \) and \( f'(x) < 0 \) for \( x > -2 \), there is a local maximum \( 4e^{-2} \) of at \( x = -2 \).

Or by using the second derivative test. Since \( f''(x) = (x^2 + 4x + 2)e^{-x} \), \( f''(0) = 2 > 0 \), so \( f(0) = 0 \) is a local minimum value. Since \( f''(-2) = -2e^{-2} < 0 \), there is a local maximum \( 4e^{-2} \) of at \( x = -2 \).

c) Using the form from (a) above, \( f'(x) = (2x + x^2)e^{-x} \), \( f''(x) = (x^2 + 4x + 2)e^{-x} \), and so \( f''(x) = 0 \), when \( x^2 + 4x + 2 = 0 \), which is \( x = \frac{-4 \pm \sqrt{8}}{2} = -2 \pm \sqrt{2} \). For \( -2 - \sqrt{2} < x < -2 + \sqrt{2} \), \( f''(x) < 0 \) and so \( f \) is concave down on that interval. For \( x < -2 - \sqrt{2} \), and \( x > -2 + \sqrt{2} \) \( f''(x) > 0 \) and so \( f \) is concave up on those intervals.

d) Since the concavity changes at the points \( x = -2 \pm \sqrt{2} \), there are points of inflection there. The actual points are \( \left(-2 \pm \sqrt{2}, (6 \mp 4\sqrt{2})e^{-2 \pm \sqrt{2}}\right) \)

7) Let \( f(x) = x^4e^{-x} \).

a) \( f'(x) = 4x^3e^{-x} - x^4e^{-x} = x^3e^{-x}(4 - x) \) \( f \) is increasing on \((0, 4)\), and decreasing on \((-\infty, 0) \cup (4, \infty)\)

b) \( f(0) = 0 \) is a relative min. \( f(4) = \frac{256}{e^4} \) is a relative max.

c) \( f''(x) = 12x^2e^{-x} - 8x^3e^{-x} + x^4e^{-x} = x^2e^{-x}(12 - 8x + x^2) = x^2e^{-x}(x - 2)(x - 6) \) \( f \) is concave up on \((-\infty, 2) \cup (6, \infty)\) and concave down on \((2, 6)\)

d) There are inflection points at \( \left(2, \frac{16}{e^2}\right) \) and \( \left(6, \frac{1296}{e^6}\right) \)
8) Suppose that \( f(x) = \frac{1}{4}x^4 - \frac{3}{2}x^3 + 3x^2 + 1 \)

a) On what intervals is the function \( f(x) \) increasing? Decreasing?

Since \( f'(x) = x^3 - \frac{9}{2}x^2 + 6x \), \( f''(x) = 0 \) when \( x^3 - \frac{9}{2}x^2 + 6x = x \left( x^2 - \frac{9}{2}x + 6 \right) \). Here is the tricky part. Since the roots to the quadratic in the parentheses are complex, (the discriminant is negative,) there are no real zeros to that piece, so \( f'(x) = 0 \), only when \( x = 0 \). Hence, there is only one stationary point at \( x = 0 \), and the sign of \( f'(x) \) only depends upon being right or left of \( x = 0 \). Since the quadratic is always positive, \( f'(x) \) is positive (negative) when \( x \) is positive (negative.)

\[
f' > 0 \Rightarrow f \text{ is increasing. Namely, } x > 0.
\]
\[
f' < 0 \Rightarrow f \text{ is decreasing. For } x < 0
\]

b) Since \( f''(x) = 3x^2 - 9x + 6 = 3(x^2 - 3x + 2) = 3(x-1)(x-2) \), \( f''(x) = 0 \) when \( x = 1, 2 \)

\[
f'' > 0 \Rightarrow f \text{ is concave up. This happens on } (-\infty, 1) \cup (2, \infty).
\]
\[
f'' < 0 \Rightarrow f \text{ is concave down. This happens on } (1, 2).
\]

c) Since the concavity changes at \( x = 1, 2 \), the points \( \left( 1, \frac{11}{4} \right) \) and \( (2, 5) \), are points of inflection.

d) What is the equation of the line tangent to the graph of \( f(x) \) at \( x = 1 \)?

Since \( f'(1) = \frac{5}{2} \), the slope of the tangent line is \( 5/2 \). The point on the line is \( \left( 1, \frac{11}{4} \right) \), so the equation of the tangent line is \( y - \frac{11}{4} = \frac{5}{2}(x-1) \).