Related Rates:

Consider the following problem:

The mechanics at a Toyota Automotive are reboring a 6-in deep cylinder to fit a new piston. The machine they are using increases the cylinder’s radius one-thousandth of an inch every 3 minutes. How rapidly is the cylinder volume increasing when the bore’s radius is 3.800 in.?

Step 1: *Picture and Variables*

Let
\[ V = \text{volume (in}^3\text{) of the bore at time } t \text{ (min)} \]
\[ r = \text{radius (in) of the bore at time } t. \]

Step 2: *Numerical Information:*

\[ r = 3.800 \text{ in and } \frac{dr}{dt} = \frac{1}{3000} \text{ in/min} \]
Step 3: What do we want to find? Ans: \( \frac{dV}{dt} \)

Step 4: How are the variables related? The cylinder’s volume is:

\[ V = (\pi r^2)h \]

Step 5: Differentiate with respect to \( t \).

\[ \frac{dV}{dt} = 12 \pi r \frac{dr}{dt} \]

Step 6: Evaluate:

\[ \frac{dV}{dt} = 12 \pi \cdot 3.800 \cdot \frac{1}{3000} \quad \text{so} \]

\[ \frac{dV}{dt} = \frac{3.8 \pi}{250} \approx 0.0478 \text{ in}^3/\text{min} \]
The method for solving related-rate problems is as follows:

1. *First draw a picture labeling all appropriate variables.* Use $t$ for time. Assume all variable are differentiable functions of time.

2. *Write down the given numerical information* (in terms of the symbols you have chosen.)

3. *Write down what you are being asked to find.* (Usually a rate, expressed as a derivative.)

4. *Write an equation that relates the variables.* You may have to combine two or more equations to get a single equation that relates the variable whose rate you want to the variables whose rates you know.

5. *Differentiate with respect to $t$.* Then express the rate you want in terms of the rate and variables whose values you know.

6. *Evaluate.* Use known values to find the unknown rate.
Example:

A spherical balloon is being filled with air at a rate such that its radius is increasing at a rate of 2 inches per second. How quickly is the volume increasing when the radius is 6 inches?

Step 1: Picture and Variables

\[ V = \text{volume (in}^3\text{)} \text{ of the balloon at time } t \text{ (s)} \]
\[ r = \text{radius (in) of the balloon at time } t. \]

Step 2: Numerical Information:

\[ r = 6 \text{ in} \quad \frac{dr}{dt} = 2 \text{ in/s} \]

Step 3: We wish to find \( \frac{dV}{dt} \).
Step 4:  *How are the variables related?*

\[ V(r) = \frac{4}{3} \pi r^3 \]

Step 5:  *Differentiate with respect to t.*

\[ \frac{d}{dt} V(t) = \frac{d}{dt} \left( \frac{4}{3} \pi [r(t)]^3 \right) = 4 \pi [r(t)]^2 \frac{dr}{dt} \]

Step 6:  *Evaluate:*

When \( r = 6 \), and \( \frac{dr}{dt} = 2 \)

\[ \frac{dV}{dt} = 4\pi (6^2)(2) = 288\pi \approx 905 \text{ in}^3/\text{s} \]
Example: A rocket is being launched. It takes off with a velocity of 550 miles per hour. 25 miles away, there is a photographer video-taping the launch. At what rate is the angle of elevation of the camera changing when the rocket achieves an altitude of 25 miles?

Step 1: Picture and Variables

Let \( h(t) \) be the height (mi) of the rocket at time \( t \) (hr).

\( \theta(t) \) is the angle of elevation at time \( t \).

Step 2: Numerical Information:

\[
\frac{dh}{dt} = 550 \text{ mi/hr} \quad h = 25 \text{ mi}
\]
Step 3: We wish to find $\frac{d\theta}{dt}$.

Step 4: How are the variables related?

$$\tan \theta = \frac{h(t)}{25}$$

Step 5: Differentiate with respect to $t$.

$$\frac{d}{dt} \tan \theta = \frac{d}{dt} \frac{h(t)}{25} = \frac{1}{25} \frac{dh}{dt}$$

or

$$\sec^2 \theta \frac{d\theta}{dt} = \frac{1}{25} \frac{dh}{dt}$$

Solving for $\frac{d\theta}{dt}$ gives

$$\frac{d\theta}{dt} = \frac{\cos^2 \theta}{25} \frac{dh}{dt}$$

Step 6: Evaluate:

$$\frac{d\theta}{dt} = 22 \cos^2 \theta$$
To determine $\cos^2 \theta$, recall that the problem asks for $\frac{d\theta}{dt}$ when the rocket is 25 mile high, so the triangle looks like:

Since $\theta = 45^\circ = \frac{\pi}{4}$, $\cos \theta = \frac{1}{\sqrt{2}}$, and hence $\cos^2 \theta = \frac{1}{2}$, and so

$$\frac{d\theta}{dt} = 22 \frac{1}{2} = 11 \text{ radians per hour.}$$
Example:

A conical tank is being filled at a rate of 9 ft$^3$/min. The tank stands point down and has a height of 10 ft and a base radius of 5 ft. How fast is the water level rising when the tank is 6 ft deep?

Step 1:  *Picture and Variables*

Let $V =$ volume (ft$^3$) of the water in tank at time $t$ (min) 
$x =$ radius (ft) of the surface of the water at time $t$. 
$y =$ depth (ft) of water and time $t$. 
Step 2:  *Numerical Information:*

\[ y = 6 \text{ ft} \quad \text{and} \quad \frac{dV}{dt} = 9 \text{ ft}^3/\text{min} \]

Step 3:  *We wish to find* \( \frac{dy}{dt} \).

Step 4:  *How the variable are related:* The water forms a cone of volume:

\[ V = \frac{1}{3} \pi x^2 y \]

Since no information is given about \( x \) or \( \frac{dx}{dt} \), we need to eliminate \( x \) from the problem. The similar triangles give

\[ \frac{x}{y} = \frac{5}{10} \quad \text{or} \quad x = \frac{1}{2} y \]

So
\[ V = \frac{1}{3} \pi \left( \frac{y}{2} \right)^2 \cdot y = \frac{\pi}{12} y^3 \]

Step 5: Differentiate with respect to \( t \).

\[ \frac{dV}{dt} = \frac{\pi}{12} (3y^2 \frac{dy}{dt}) = \frac{\pi}{4} (y^2 \frac{dy}{dt}) \]

Step 6: Evaluate:

\[ 9 = \frac{\pi}{4} (6^2 \frac{dy}{dt}) \quad \text{so} \]

\[ \frac{dy}{dt} \bigg|_{t} = \frac{1}{\pi} \approx 0.32 \text{ ft/min.} \]