Assume the same problem description as in Lecture 23, "Introduction to the Ball Walk". Much of the material on estimating the volume of a convex body can be found in [1], [2], and [3].

In the last class we began to study the speedy walk where given a point \( x \in K \) the transition probability to a set \( A \) is defined by

\[
P(x, A) = \int_{y \in B(x, \delta) \cap K \cap A} \frac{dy}{Vol_n(B(x, \delta) \cap K)}.
\]

In the typical case we will set \( \delta \leq 1/\sqrt{n} \). It was shown that the stationary distribution \( \mu \) of this Markov chain is given by

\[
\mu(A) = \frac{1}{L} \int_A l(x) dx
\]

where \( l(x) \) is the local conductance of the point \( x \in K \) and is defined by

\[
l(x) = \frac{Vol_n(B(x, \delta) \cap K)}{Vol_n(B(x, \delta))}.
\]

It is possible for us to modify our sampling procedure in order to ensure that in the limit we will obtain uniform samples from \( K \). Assume that \( l(x) \geq c > 0 \) for all \( x \in K \). Now

(i) Generate a sample according to \( \mu \).

(ii) Accept each sample point \( x \) with probability \( c/l(x) \).

The resulting sampling is uniform since

\[
P(\text{sample point } x \in K) \propto l(x) \cdot \frac{c}{l(x)} = c.
\]

One way to guarantee that \( l(x) \) is sufficiently large is to make some sort of assumption on the boundary of \( K \).

**Lemma 24.1.** [1] If for each \( x \in K \) there exists a \( y \in B(x, \delta) \cap K \) such that \( B(y, \delta) \subseteq K \) where \( \delta \leq c/\sqrt{n} \), then \( 4 \leq l(x) \leq 1 \).

However, the conditions in the lemma are fairly strong so a weaker requirement would be nice. Observe that points fairly far from the boundary of \( K \) will be sampled roughly uniformly, and only points near the boundary are heavily non-uniform. This can be exploited by scaling \( K \) down by a small factor, only accepting Speedy when it samples in this fairly uniform region, and then scaling the result back out until it entirely covers \( K \).

**Theorem 24.2.** [3] Suppose that the Speedy chain is run long enough that the variation distance is at most \( \delta \). Then, given a sample \( v \in K \) from the Speedy chain check if \( \frac{2n}{4n-1} v \in K \); if it is then return \( \frac{2n}{4n-1} v \) as a sample from \( K \), otherwise run Speedy again and repeat this procedure. Then if

\[
\delta \leq 1/\sqrt{8n \log(n/\varepsilon)}
\]

then the final sample is within \( 10 \varepsilon \) from being uniform.
We now must show that the speedy walk is rapidly mixing. Recall the following from last class for a continuous space chain.

**Theorem 24.3.**

\[ \tau(1/4) \leq 15000 \left[ \int_{\pi_1}^{1/2} \frac{dx}{x(\Phi(x))^2} + \frac{1}{\Phi} \right] \]

where \( \pi_1 = \sup \{ t : \forall A \subseteq \Omega \text{ s.t. } \pi(A) = t, P(x, A^C) \geq 1/10 \forall x \in A \} \).

This can be used to study \( \tau(\varepsilon) \) by a previous inequality from lecture seven.

**Lemma 24.4.**

\[ \tau(\varepsilon) \leq \tau(\delta) \left\lceil \log_2 \left( \frac{1}{\Phi(1/2\varepsilon)} \right) \right\rceil. \]

Thus \( \tau(\varepsilon) \leq \tau(1/4) \left\lceil \log_2 (1/2\varepsilon) \right\rceil \).

We can bound \( \pi_1 \) from below for the speedy walk, so long as the step size \( \delta \) is sufficiently small. Let \( D \) be the diameter of \( K \).

**Lemma 24.5.** \( \pi_1 \leq (1/2)(\delta/D)^{2n} \) if \( \delta \leq 1 \).

**Proof.** Let \( x \in K \) and \( S \subseteq K \) be such that \( \pi(S) \leq (1/2)(\delta/D)^{2n} \). It suffices to show that \( P_x(K \setminus S) \geq 1/10. \) Blowing up \( K \cap B(x, \delta) \) by a factor of \( D/\delta \) covers \( K \) so that

\[
\begin{align*}
l(x) &= \frac{Vol_n(B(x, \delta) \cap K)}{Vol_n(B(x, \delta))} \\
&\geq \left( \frac{\delta}{D} \right)^n \frac{Vol_n K}{Vol_n(B(x, \delta))}.
\end{align*}
\]

Using this inequality and the fact that \( l(x) \leq 1 \) we get

\[
\begin{align*}
\pi(S) &= \frac{\int_S l(x) dx}{\int_K l(x) dx} \\
&\geq \left( \frac{\delta}{D} \right)^n \frac{Vol_n K}{Vol_n(B(x, \delta))} \frac{Vol_n S}{Vol_n K}.
\end{align*}
\]

Thus

\[
\begin{align*}
P_x(S) &= \frac{Vol_n(S \cap B(x, \delta))}{Vol_n(K \cap B(x, \delta))} \\
&\leq \frac{Vol_n(S)}{l(x)Vol_n(B(x, \delta))} \\
&\leq \left( \frac{\delta}{D} \right)^n \pi(S) \frac{Vol_n(B(x, \delta))}{Vol_n(K)} \\
&= \pi(S) \left( \frac{D}{\delta} \right)^{2n} \frac{Vol_n(B(x, \delta))}{Vol_n(K)} \\
&\leq 1/2.
\end{align*}
\]

\[ \square \]
References

