\[ \pi_1 \geq \frac{1}{2} \left( \frac{\delta}{D} \right)^{2n} \]

\[ \Phi(x) \geq \min \left\{ \frac{1}{288\sqrt{n}}, \frac{\delta}{81\sqrt{n}D} \ln(1 + \frac{1}{x}) \right\} \]

\[ = \begin{cases} \frac{1}{288\sqrt{n}} & \text{if } x \leq e^{-D/38} \\ \frac{\delta}{81\sqrt{n}D} \ln(1 + \frac{1}{x}) & \text{if } x \geq e^{-D/38} \end{cases} \]

Then

\[ \tau(\frac{1}{4}) \leq 15000 \left[ 288^2n \int_{\frac{e^{-D/38}}{2}}^{e^{-D/38}} \frac{dx}{x} + 81^2n \left( \frac{D}{\delta} \right)^2 \int_{e^{-D/38}}^{\frac{1}{2}} \frac{dx}{x \ln^2 1/x} \right] \]

\[ = c_1 n \ln x \left[ \frac{e^{-D/38}}{2} \right] + c_2 n \left( \frac{D}{\delta} \right)^2 \ln 2 \]

\[ = O(n^2 \ln \frac{D}{\delta} + n \left( \frac{D}{\delta} \right)^2) \]

\[ = O(n^2) \quad \because D = O(\sqrt{n}), \quad \delta = \Omega \left( \frac{1}{\sqrt{n}} \right) \]

This is as good as possible, given fixed \( \delta \) and \( D \). Consider cube of side length \( D \). Diagonal is length \( D\sqrt{n} \).

Takes \( \frac{D\sqrt{n}}{4} \) steps to go from one corner to the opposite. About \( \left( \frac{1}{2} \frac{D\sqrt{n}}{4} \right)^2 \) steps to go halfway across the cube, same as what we showed.

It remains to bound

\[ \max_{\tau(A) \leq \tau(A)} \frac{Q(A, A^c)}{\pi(A)} \]

Basic idea: Want to bound \( Q(A, A^c) = \frac{1}{2} [Q(A, A^c) + Q(A^c, A)] \). Let

\[ S_1 = \{ x \in A | P_x(A^c) \leq \frac{1}{16} \} \]

\[ S_2 = \{ x \in A^c | P_x(A) \leq \frac{1}{16} \} \]

\[ B = K \setminus (S_1 \cup S_2) \]
Then
\[ Q(A, A^c) \geq \frac{1}{2} \frac{\text{Vol}_n(B)}{16}. \]

It suffices to find how big $B$ is.

**Lemma 25.1. Localization lemma (Lovasz and Simonovitz)** Let $g$ and $h$ be lower semicontinuous (limits of monotone increasing sequence of continuous functions, e.g. indicators of open sets, \[ \int_{\mathbb{R}^n} g(x)dx > 0 \text{ and } \int_{\mathbb{R}^n} h(x)dx = 0. \]

Then there are two points $a, b \in \mathbb{R}^n$ and a linear function $l : [0, 1] \to \mathbb{R}_+$ exist such that

\[ \int_0^1 l(t)^{n-1} g((1 - t)a + tb)dt > 0 \text{ and } \int_0^1 l(t)^{n-1} h((1 - t)a + tb)dt = 0. \]

3d cross sectional area $l(t)^2$. Reduced to needle-like case.

Method of applying: Want to show fact in $n$-dim. Assume counterexample, writes to integrals. Localization reduces to one dimension. Show impossible in one-dimension.

Sketch of Proof: Suffices to assume $g$ and $h$ are continuous because if $g = \lim_{k \to \infty} g_k$ then $\int g = \lim \inf g_k$ (Monotone convergence), and likewise for $h$. To get the result, use a bisection argument (Ham Sandwich)

Either
\[ \int_{H_1} g \geq \frac{1}{2} \int_{\mathbb{R}^2} g \text{ or } \int_{H_2} g \geq \frac{1}{2} \int_{\mathbb{R}^2} g. \]
Let $K_1$ denote the appropriate half-space, i.e.,

$$\int_{K_1} g(x) dx > 0 \quad \text{and} \quad \int_{K_1} h(x) dx > 0.$$ 

Now consider all rational points (countably many) put in a list and bisecting down the list constructing sequence $K_0 \supseteq K_1 \supseteq \ldots$. Then $\bigcap K_i$ has dimension 0 or 1.