Lecture 6 - The mixing time of simple random walk on a cycle

Friday, August 27

To date the only Markov chain for which we know much about the mixing time is the walk on the uniform two-point space. Today we use Theorem 2 of the previous lecture to find the mixing time of a non-trivial Markov chain.

Consider the simple random walk on the cycle $C_\ell$ (equivalently, on $\mathbb{Z}_\ell$). This has transition matrix

$$P(i, j) = \begin{cases} \frac{1}{\ell} & \text{if } j \equiv i \pm 1 \mod \ell \\ 0 & \text{otherwise} \end{cases}$$

Fix some point on the cycle and label it as 0, then number the remaining points in the clockwise direction up to $\ell - 1$.

If $\ell$ is even then this walk is periodic and the Markov chain does not converge, so let us restrict our attention to the case when $\ell$ is odd. In this case the eigenvalue / eigenvector pairs are given by

<table>
<thead>
<tr>
<th>eigenvalue</th>
<th>eigenvector(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1$</td>
<td>constant</td>
</tr>
<tr>
<td>$\frac{1}{2}$</td>
<td>$\cos\left(\frac{2\pi j}{\ell}\right)$, $\sin\left(\frac{2\pi j}{\ell}\right)$, $k$</td>
</tr>
</tbody>
</table>

where $0 \leq k \leq \ell - 1$ is the position around the cycle. This is easily verified to be a valid eigenbasis.

Now, in order to apply Theorem 2 we require orthonormal eigenvectors of the matrix given by $S_{ij} = \sqrt{\frac{\pi(i)}{\pi(j)}} P_{ij}$. But, $\pi$ is uniform and so this reduces to $S = P$ and the eigenvectors given above for $P$ will suffice. In general, if $\pi$ is uniform then $S = P$.

It remains to make these orthonormal. Clearly the constant eigenvector should be $\vec{u}_1 = 1/\sqrt{\ell}$. Also, eigenvectors with different eigenvalues are always orthogonal.

Now, consider the eigenvector $\vec{v} = \cos\left(\frac{2\pi j}{\ell}\right)$. This has norm

$$\vec{v} \cdot \vec{v} = \sum_{k=0}^{\ell-1} \cos^2\left(\frac{2\pi j}{\ell} k\right) = \sum_{k=0}^{\ell-1} \frac{1 + \cos\left(\frac{4\pi j}{\ell} k\right)}{2} = \frac{\ell}{2} + \frac{1}{2} \sum_{k=0}^{\ell-1} \cos\left(\frac{4\pi j}{\ell} k\right) = \frac{\ell}{2} + \frac{1}{2} \Re\left(\sum_{k=0}^{\ell-1} \exp\left(i \frac{4\pi j}{\ell} k\right)\right)$$

where the second equality used the identity $\cos^2 x = \frac{1 + \cos 2x}{2}$ and $\Re(x)$ denotes the real part of $x$. If $4j$ and $\ell$ are relatively prime then this is a sum over the $\ell$-th roots of unity $\xi_1, \xi_2, \ldots, \xi_\ell$. But this is just zero if $\ell > 1$, because

$$x^\ell - 1 = (x - \xi_1)(x - \xi_2) \cdots (x - \xi_\ell) = x^\ell - \left(\sum_{i=1}^{\ell} \xi_i\right) x^{\ell-1} + \cdots + (-1)^\ell \prod_{i=1}^{\ell} \xi_i$$

and the coefficient of $x^{\ell-1}$ is zero if $\ell - 1 > 0$. If $4j$ and $\ell$ are not relatively prime then the sum is $(4j, \ell)$ times the sum of the $\frac{\ell}{(4j, \ell)}$ roots of unity, and everything still adds to zero. In short, $\vec{v} \cdot \vec{v} = \ell/2$.

A similar argument holds for the sinusoidal eigenvectors, using the identity $\sin^2 x = \frac{1 - \cos 2x}{2}$.
Finally, eigenvectors with the same eigenvalue satisfy
\[ \sum_{k=0}^{\ell-1} \sin \left( \frac{2\pi j k}{\ell} \right) \cos \left( \frac{2\pi j k}{\ell} \right) = \frac{1}{2} \sum_{k=0}^{\ell-1} \sin \left( \frac{4\pi j k}{\ell} \right) = \frac{1}{2} \Im \left( \sum_{k=0}^{\ell-1} \exp \left( \frac{4\pi j k}{\ell} \right) \right) = 0, \]
where \( \Im \) denotes the imaginary part of the sum of roots of unity.

The orthonormal eigenbasis is then the basis given earlier, normalized by a factor of \( \sqrt{2/\ell} \).

Finally the \( L^2 \) distance can be determined.

\[
\left\| 1 - \frac{P^j(x, \cdot)}{\pi(x)} \right\|_{2, \pi}^2 \leq \sum_{m=1}^{n} \frac{\lambda_m^j (u_m)^2}{\pi(x)} - 1
\]

\[
= \frac{1}{1/\ell} \sum_{j=1}^{(\ell-1)/2} \cos^2 \left( \frac{2\pi j}{\ell} \right) \left( \frac{2}{\ell} \cos^2 \left( \frac{2\pi j k}{\ell} \right) + \sin^2 \left( \frac{2\pi j k}{\ell} \right) \right)
\]

\[
= 2 \sum_{j=1}^{(\ell-1)/2} \cos^2 \left( \frac{2\pi j}{\ell} \right) + 2 (\ell-1)/2 \sum_{j=1+\ell}^{(\ell-1)/2} \cos^2 \left( \frac{2\pi j}{\ell} \right)
\]

\[
= 2 \sum_{j=1}^{(\ell-1)/2} \cos^2 \left( \frac{\pi j}{\ell} \right) = \left\| 1 - \frac{P^j(x, \cdot)}{\pi(x)} \right\|_{2, \pi}^2
\]

The final equality applied the identity \( \cos(\pi(1-x)) = -\cos(\pi x) \) to the second sum.

This is our first non-trivial example where we could determine a distance exactly. However, in its current form this is not particularly useful since we have no notion of how large the sum is. We now simplify this via a procedure suggested in Diaconis’ book [1].

Observe that \( \cos x \leq e^{-x^2/2} \) when \( x \in [0, \pi/2] \). This follows by letting \( h(x) = \log(e^{-x^2/2} \cos x) \), then \( h'(x) = x - \tan x \leq 0 \) for \( x \in [0, \pi/2] \) and so \( h(x) \leq h(0) = 0 \). It follows that

\[
\sum_{j=1}^{(\ell-1)/2} \cos^2 \left( \frac{\pi j}{\ell} \right) \leq \sum_{j=1}^{(\ell-1)/2} e^{-\pi^2 j^2 / \ell^2}
\]

\[
\leq e^{-\pi^2 / \ell^2} \sum_{j=1}^{\infty} e^{-\pi^2 (j^2 - 1) t / \ell^2}
\]

\[
\leq e^{-\pi^2 / \ell^2} \sum_{j=0}^{\infty} e^{-3\pi^2 j t / \ell^2}
\]

\[
= e^{-\pi^2 / \ell^2} \frac{1}{1 - e^{-3\pi^2 t / \ell^2}}
\]

where the first inequality applied \( \cos x \leq e^{-x^2/2} \), the second factored out a term and extended the sum, the third used the inequality \( j^2 - 1 \geq 3(j-1) \) for \( j \in \mathbb{Z}_{\geq 0} \) (the case \( j = 1 \) is trivial, and \( j \geq 2 \) is simple algebra), and the final equality is because this was a geometric series.

For a final simplification observe that if \( t \geq \frac{\ell^2}{8\pi^2} \log 2 \approx \frac{\ell^2}{60} \) then \( e^{-3\pi^2 t / \ell^2} \leq 1/2 \). Therefore

\[
\left\| P^j(x, \cdot) - \pi \right\|_{TV} \leq \frac{1}{2} \left\| 1 - \frac{P^j(x, \cdot)}{\pi} \right\|_{2, \pi} \leq e^{-\pi^2 t / 2\ell^2} \leq e^{-\pi^2 t / 40}
\]

To finish this off, solve \( e^{-\pi^2 t / \ell^2} \leq \epsilon \) to find that

\[
\frac{\tau}{\pi^2} \leq \frac{2\ell^2}{\pi^2} \log \frac{1}{\epsilon} \quad \text{if } \epsilon \leq e^{-\pi^2 / 80} \approx 0.88
\]
In the next lecture we show a matching lower bound, showing that this upper bound is essentially correct.

References