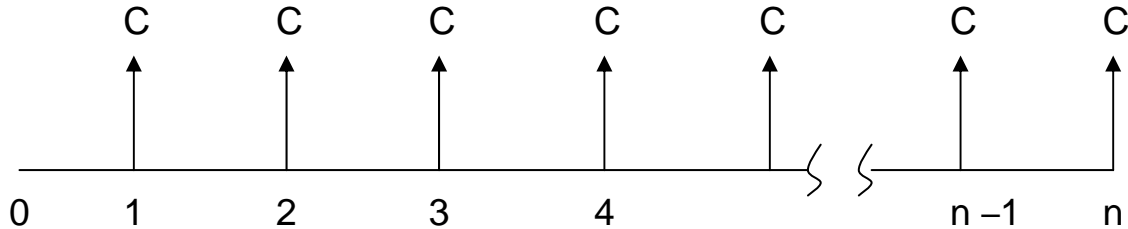## Derivation of Annuity and Perpetuity Formulae

A. Present Value of an Annuity (Deferred Payment or Ordinary Annuity)

We have in the shown in the lecture notes and in "Compounding and Discounting" that the present value of a set of general cash flows is:

$$PV = \sum_{t=0}^{n} \frac{C_t}{(1+r)^t} \qquad \text{or} \qquad PV = \sum_{t=0}^{n} C_t (1+r)^{-t}$$

The cash flow diagram above shows a set of cash flows between $t = 1$ and $t = n$ where each cash flow is identical. This is called an annuity. Since there is no cash flow at $t = 0$ and the rest of the cash flows are identical, that is:

$$C_0 = 0 \text{ and } C_1 = C_2 = C_3 = \ldots C_n,$$

we can simplify a bit, and in particular, factor out the constant cash flow C:

$$PV = C \sum_{t=1}^{n} (1+r)^{-t}$$

We can also write this as follows:

$$PV = \frac{C}{(1+r)} + \frac{C}{(1+r)^2} + \ldots \frac{C}{(1+r)^n}$$

The objective is to combine this set of finite terms into one compact equation. We will make a substitution for the quantity $1/(1 + r)$ to simplify the notation that follows:

Let $u = \frac{1}{1+r}$, then $PV = Cu + Cu^2 + \ldots Cu^n$

Multiply each side by u:

$$uPV = Cu^2 + Cu^3 + \ldots Cu^n + Cu^{n+1}$$

Subtract the second equation from the first:

$$PV = Cu + Cu^2 + ...Cu^n$$
$$- uPV = \quad\quad Cu^2 + ...Cu^n + Cu^{n+1}$$

$$\overline{PV - uPV = Cu.............................- Cu^{n+1}}$$

Notice that in the subtraction all the middle terms were eliminated, leaving us with only two terms instead of n terms. Solving for PV:

$$PV(1-u) = Cu - Cu^{n+1}$$

$$PV = \frac{Cu - Cu^{n+1}}{1-u} = \frac{Cu(1-u^{n)}}{1-u}$$

Substitute back $u = \dfrac{1}{1+r}$:

$$PV = \frac{C\left(\dfrac{1}{1+r}\right)\left(1 - \dfrac{1}{(1+r)^n}\right)}{\left(1 - \dfrac{1}{(1+r)}\right)}$$

Multiply the numerator and the denominator by $1 + r$:

$$PV = \frac{C\{1-(1+r)^{-n})\}}{1+r-1} \quad\quad \text{or} \quad\quad PV = C\left[\frac{1-(1+r)^{-n}}{r}\right]$$

The number in the brackets is called the annuity discount factor.

The derivation above has shown that the present value of an annuity can be calculated as:

$$PV = \sum_{t=1}^{n} C(1+r)^{-t} \quad\quad \text{or as} \quad\quad PV = C\left[\frac{1-(1+r)^{-n}}{r}\right]$$

But the first form has n terms, while the second is very compact and is the product of the cash flow and the annuity discount function formula. From the derivation it is also very clear that the annuity discount factor is the sum of the individual discount factors for each cash flow. It should be easy to show that the annuity factor must be a number less than n.

B. Future Value of an Annuity

Since the future value (FV) for an amount worth PV at the present time is:

$$FV_n = PV(1+r)^n$$

and for an annuity, the present value was just shown in Part A to be:

$$PV = C\left[\frac{1-(1+r)^{-n}}{r}\right]$$

Then we can combine these two formulas to give us the future value of an annuity:

$$FV_n = C\left[\frac{1-(1+r)^{-n}}{r}\right](1+r)^n$$

Simplifying:

$$FV_n = C\left[\frac{(1+r)^n - \dfrac{(1+r)^n}{(1+r)^n}}{r}\right]$$

$$FV_n = C\left[\frac{(1+r)^n - 1}{r}\right]$$

C. Present Value of a Perpetuity

An annuity which has infinite terms ($n = \infty$) is called a perpetuity.

To derive the formula for a perpetuity, we use the following property:

$$\text{When} \quad r > 0$$

$$\lim_{n \to \infty}(1 + r)^{-n} = 0$$

We can demonstrate this using a calculator. Trying $n = 100$, and r = .10 we get 0.00007257. When we increase n to be 1000, we get 4.04892 x 10^{-42}, which is a very small number. Try this yourself!

Now go back to the present value formula for the annuity:

$$PV_{Annuity} = C\left[\frac{1 - (1 + r)^{-n}}{r}\right]$$

For a perpetuity, we use the same formula but n will approach infinity!

$$PV_{Perpetuity} = \lim_{n \to \infty} C\left[\frac{1 - (1 + r)^{-n}}{r}\right]$$

Using the limit equation we have demonstrated above, the term with the negative n exponent goes to zero as n goes to infinity, and our perpetuity equation is reduced to a very simple:

$$PV_{Perpetuity} = \frac{C}{r}$$

D. Present Value of an Annuity with Growing Payments

Occasionally, we need to calculate the present value of a set of payments where C is not constant. If C grows at a constant rate g, such that $C_i = C_{i-1}(1+g)$, we would call it a growing payment annuity.

An example of this is if dividends at time t = 0 are \$10, and grow at a rate of 5% every period. That is, D_0 = \$10, D_1 = 10(1.05) = \$10.50 and D_2 = $10(1.05)^2$ = \$11.025.

If this continues for n periods, we can get an expression for the present value, and it is also very simple to calculate if n approaches infinity. To show this, we use some of the results we obtained in our annuity derivation for a constant payment C.

Using the substitution $u = \dfrac{1}{1+r}$,

we were able to show that

$$PV = Cu + Cu^2 + \ldots\ldots Cu^n$$

and that:

$$PV = \frac{Cu(1-u^n)}{1-u}$$

If we have a growing payment annuity where $C_i = C_{i-1}(1+g)$, and g = rate of growth, our present value will be:

$$PV = \frac{C_0(1+g)}{(1+r)} + \frac{C_0(1+g)^2}{(1+r)^2} + \ldots \frac{C_0(1+g)^3}{(1+r)^n}$$

Notice that our first cash flow is at t = 1 although we use C_0 in our formula. That is,

$C_1 = C_0(1+g),$ $C_2 = C_1(1+g)$ or $C_2 = C_0(1+g)^2$ etc.

Our substitution for the growing payment annuity will be $u = \dfrac{1+g}{1+r}$, which yields the familiar equation:

$$PV = C_0 + C_0 u^2 + \ldots\ldots C_0 u^n$$

Since this equation is exactly what we had before, we can use our prior result that this will equal:

$$PV = \frac{C_0 u(1 - u^n)}{1 - u}$$

The only difference is that we need to substitute back our new definition for u:

$$u = \frac{1 + g}{1 + r}.$$

The manipulation to simplify is a bit complicated, so I will provide the results and the show the details:

$$PV = C_1 \left[\frac{1 - \left(\dfrac{1 + g}{1 + r}\right)^{-n}}{r - g} \right]$$

Details:

$$PV = \frac{C_0 u(1 - u^n)}{1 - u} \qquad\qquad u = \frac{1 + g}{1 + r}$$

$$PV = \frac{C_0 \left(\dfrac{1 + g}{1 + r}\right)\left[1 - \left(\dfrac{1 + g}{1 + r}\right)^n\right]}{1 - \left(\dfrac{1 + g}{1 + r}\right)}$$

Now multiply both the numerator and the denominator by $1 + r$:

$$PV = \frac{C_0(1 + g)\left[1 - \left(\dfrac{1 + g}{1 + r}\right)^n\right]}{(1 + r) - (1 + g)} = \frac{C_0(1 + g)\left[1 - \left(\dfrac{1 + g}{1 + r}\right)^n\right]}{r - g}$$

Since $C_1 = C_0(1 + g)$

$$PV = \frac{C_1\left[1-\left(\dfrac{1+g}{1+r}\right)^n\right]}{r-g}$$

One application for the constant growth annuity formula is when dividends grow at a constant rate g for n periods. Even more popular is the assumption that dividends continue to grow forever at this rate. It is assumed that your first dividend is at t = 1.

You want to calculate the fair price for the stock at t = 0, or P_0. This is the present value of all future dividends which will start to grow at t = 0 using rate g, but your first dividend is D_1. Since stocks are risky, you will discount using a rate higher than the risk –free rate. It is common to use the expected rate of return for the stock for this discount rate, and we will denote this rate here using k.

$$P_0 = \frac{D_1\left[1-\left(\dfrac{1+g}{1+k}\right)^n\right]}{k-g}$$

If n approaches infinity, as long as the discount rate is greater than the growth rate or g < k:

$$\lim_{n\to\infty}\left(\frac{1+g}{1+k}\right)^n = 0$$

To show this try $g = 0.05$, $k = 0.10$, and $n = 10{,}000$ on your calculator.

The above *Dividend Discount Model* will then reduce to the compact equation shown below:

$$P_0 = \frac{D_1}{k-g}$$

This is also called the *Gordon Growth Model*.