Portfolio Theory and the Capital Asset Pricing Model (CAPM)

A natural consequence of investing in risky securities is the need to deal with uncertain cash flows. One objective of portfolio theory is the identification of the important attributes of risky assets, particularly when individual assets are combined in a single portfolio. Another objective is the determination of the optimal portfolio for investors. The Capital Asset Pricing Model (CAPM) builds on portfolio theory, but goes one step further by identifying the important attributes of risky assets in the determination of their expected return.

The most important results are easy to explain in a few words, provided that the statistical concepts of mean, variance, and covariance is understood. Incidentally, the mean value is often called the expected value, and the square root of the variance is the standard deviation. We start by assuming that the only attributes of risk assets, such as stocks or bonds, that investors care about is the expected return and the standard deviation of returns, which is a proxy for risk.   But this assumes that only a single asset is being held. A portfolio is a combination of single assets, and it follows that holders of portfolios will care only about the expected return and standard deviation of the portfolio they are holding.

It will be discussed that under the usual circumstances, everyone should be holding well diversified portfolios. With respect to individual assets, an investor will care primarily about its contribution to the risk of their portfolio. Although the risk of an asset when held alone is measured by its standard deviation of returns, its contribution to the risk of a portfolio is measured by its covariance with a hypothetical portfolio we call the market portfolio. The beta of a risky asset is essentially a slightly altered measure of this covariance.   Because beta is the attribute of risky assets that matters in a portfolio context, beta will be the asset specific variable to determine the expected return of the asset.

If you are not familiar with the above terms, they will be defined below. Although the summary of portfolio theory is simple, it requires equations and diagrams to explain the reasoning behind it. The advantage of equations is that they allow you to write statements in a very exact and compact manner. The advantage of diagrams is that they help you to visualize. Portfolio theory was developed by Harry Markowitz 50 years ago, and to celebrate this event, Professor Mark Rubinstein (University of California - Berkeley) has published a short fifty-year retrospective on the topic. Although his three pages are almost devoid of equations and diagrams, Rubinstein makes the point that it was the mathematical approach which lead to a formalization of the diversification benefits of portfolio selection. The theory provided a number of non-intuitive results which ran contrary to the wisdom of earlier works which were based more on intuition than analysis.   As Rubinstein concludes the essay, "Markowitz can boast that he found the field of finance awash in the imprecision of English and left it with the scientific precision and insight only made possible by mathematics."[1]

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[1] Mark Rubinstein, "Markowitz's "Portfolio Selection": A Fifty-Year Retrospective, *Journal of Finance* 57, No. 3 (June 2002), pp. 1041-1045 (Copy available from your instructor)

The objective of this document is to help you put together the various sources you have obtain from me on this topic. Most of the material will not be new, but will represent the oral discussions I present along with my equations, diagrams and numerical examples. To test your understanding of the material, you should answer the questions I pose in the accompanying document: "Portfolio Theory Discussion Questions".[2]

Risky Stock Returns

Stocks are risky assets, where future dividends and selling price are uncertain. We will represent possible returns on stock i with the symbol R_i. Consider a single period over which the population of n possible returns and their associated probabilities are available. We can calculate several summary measures of the distribution of possible stock returns:

mean or expected return
$$\overline{R_i} = \sum R_i P_i$$

variance
$$var_i = \sigma_i^2 = \sum (R_i - \overline{R_i})^2 P_i$$

standard deviation
$$\sigma_i = \sqrt{\sigma_i^2}$$

These are just the standard summary measures of a random variable that you might have run across in an introductory statistics course. For us, the random variable is R_i, the return on a stock. A number of different symbols and terms are used in practice, and it is best to learn all of them. We also vary them in this document, primarily for space and style considerations.

The expected value or mean value is the weighted average of all possible outcomes, where the weights are the probabilities of the outcome.[3] It measures the central tendency of the distribution. The variance is also an expected value. It is the expected value of the squared differences between the possible outcomes and the mean return. We need to square the differences; otherwise negative possible outcomes will cancel positive outcomes in a symmetrical distribution. The standard deviation is just the square root of the variance, and provides a measure with the same scale as the original random variable. It is common to use the variance, or alternatively, the standard deviation, as a measure of the risk.

For stock returns, it can easily be accepted that investors only care about the mean and standard deviation of a risky stock, because the normal distribution, the familiar family of bell shaped distributions, is often used to describe the probability distribution of stock returns. The normal distribution is completely described by the mean and the standard deviation. In addition, the normal distribution is symmetric, which solves the

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[2] Answers are available in "Portfolio Theory Discussion Question Answers".
[3] To simplify the notation we suppress the index for the n possible outcomes.

problem that investors are only concerned with downside risk, while the variance considers deviations on either side of the mean. With a symmetric distribution this is not an issue.

Portfolios

A portfolio is a set of individual assets held by an entity. We will first consider a portfolio of two risky stocks. If investors care about the mean and standard deviation of an individual asset, they will care about the mean and standard deviation of the return of a portfolio if they are holding a portfolio. Suppose that x_i is the proportion invested in stock i and x_j is the proportion invested in stock j.[4] For any possible pair of outcomes that can occur for stock i and stock j, the portfolio return would be:

$$R_p = x_i R_i + x_j R_j$$

If we have all the possible outcomes for each stock and the associated joint probabilities for each occurrence, we could calculate the portfolio return using the above equation and then continue to use our previous summary measures for the portfolio mean and standard deviation. There is another approach that is more practical both for our analysis and also in practice. If we have the portfolio proportions and the summary measures for each individual asset, plus one additional measure called the correlation coefficient, we can also use the following two formulas to obtain the portfolio mean and standard deviation:

portfolio mean $\qquad\qquad\qquad \overline{R}_p = x_i \overline{R}_i + x_j \overline{R}_j$

portfolio standard deviation $\qquad \sigma_p = \sqrt{x_i^2 \sigma_i^2 + x_j^2 \sigma_j^2 + 2 x_i x_j \rho_{ij} \sigma_i \sigma_j}$

We now explain the correlation coefficient ρ_{ij}.[5] When you have two random variables, there exists a summary measure called the covariance, which captures how the variables move together:

covariance $\qquad \text{cov}_{ij} = \sigma_{ij} = \sum (R_i - \overline{R}_i)(R_j - \overline{R}_j) P_{ij}$

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[4] For a two stock portfolio, the proportions must add up to 1.0. For example, if you invest 75% in asset i, $x_i = .75$ and $x_j = .25$.
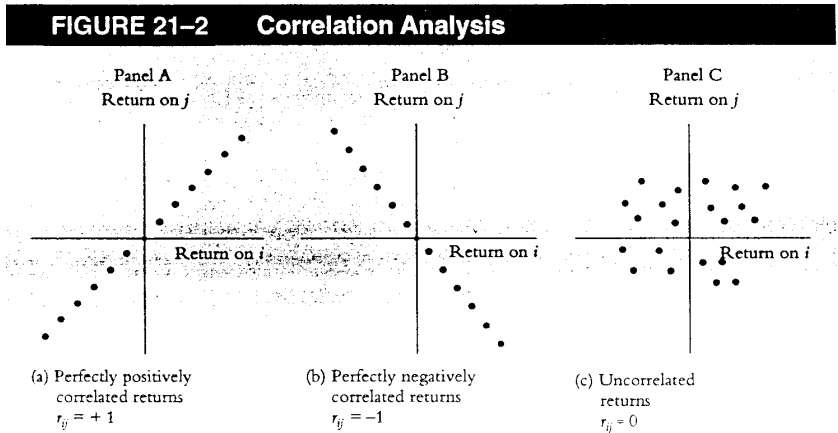
[5] The standard deviation of stock i is often represented by the Greek letter sigma, σ_i. The variance might be represented by σ_i^2 or σ_{ii}. This will become clear after you see that the covariance is represented by σ_{ij}, and that a covariance of a stock return with itself is nothing more than the variance of that stock. The population correlation coefficient is usually represented by the Greek letter rho, ρ_{ij}.

This measure is also an expected value. It is the weighted average of the product of the deviation of the return on stock i from its mean and stock j from its mean. If the tendency of the relationship between the two stocks is such that stock i and stock j vary together, the products will tend toward positive numbers and the resulting covariance will be positive. If the return on stock i is mostly above its mean when the return on stock j is below its mean, or the other way around, then the covariance is a negative number.[6]

Although the covariance can take on any value, the magnitude of the measure is not meaningful. This is explained with the story of calculating the covariance of the height of fathers with their sons. If height is measured in inches, the covariance will be greater than if we measure height in feet, but the relationship between the two will not have changed! We standardize the covariance measure by dividing it by both individual standard deviations. The resulting measure is called the correlation coefficient.

correlation coefficient $$\rho_{ij} = \frac{\text{cov}_{ij}}{\sigma_i \sigma_j}$$

The correlation takes on a value between –1 and +1, where the magnitude is now meaningful and should not depend on our scale. $\rho_{ij} = 1$ means that the two stocks vary perfectly together, while $\rho_{ij} = -1$ means they are perfectly negatively correlated. $\rho_{ij} = 0$ indicates no relationship between the stock returns. These extreme measures are demonstrated in the figure below:

FIGURE 21–2   Correlation Analysis

Panel A	Panel B	Panel C
Return on j	Return on j	Return on j

(a) Perfectly positively correlated returns $r_{ij} = +1$

(b) Perfectly negatively correlated returns $r_{ij} = -1$

(c) Uncorrelated returns $r_{ij} = 0$

Why is the covariance important? It plays no part in determining the expected return of the portfolio but it is quite important in a part in determining the standard deviation of a portfolio. In the next section we introduce a very important tool to help us visualize what happens when we combine individual stocks. In particular, we will focus on what happens as the correlation between the stocks vary (a property of the stocks we select) and as we vary the proportion invested in each (a variable we can control). The

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[6] It is useful to understand why this is so by considering the possible signs of each of the product.

tool is a graph using the standard deviation on the horizontal axis, and expected return on the vertical axis. For these graphs, any point can represent either a stock or a portfolio of stocks and the coordinates tell us the expected return and the standard deviation of the returns.

Combining Individual Stocks to Create a Portfolio

Let us repeat our two important formulas:

portfolio mean
$$\overline{R}_p = x_i \overline{R}_i + x_j \overline{R}_j$$

portfolio standard deviation
$$\sigma_p = \sqrt{x_i^2 \sigma_i^2 + x_j^2 \sigma_j^2 + 2 x_i x_j \rho_{ij} \sigma_i \sigma_j}$$

Below we only show the possibilities for 3 correlation coefficients: -1, 1, and 0.2. The end points of the graph are for allocating 100% of your portfolio in only one stock, and as you start investing in the other, you move along the graph indicated.
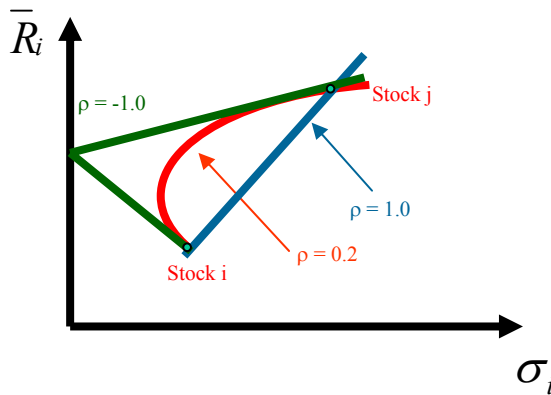
Consider first combining stock i and j when $\rho = 1$. If you are 100% in stock i, you are at the point indicated by stock i. As you increase the proportion of your investment in stock j, you will move along a straight line toward it. Your portfolio risk, as well as your portfolio expected return will increase in a linear fashion. But this is not true for the relationship when $\rho = -0.2$. You actually start to decrease risk (move to the left) even while you are increasing expected return (moving upward). When $\rho = -1$, you can actually find a mixture of both stocks that reduces all your risk while still giving you an increase in expected return over stock i alone.

Extending this analysis to cover multiple stocks in a single portfolio, we end up with what we call the feasible set. The picture below only shows a few stocks, but with only a few actual stocks one can form portfolios which will reach any point in the area below the curve.

The Feasible Set
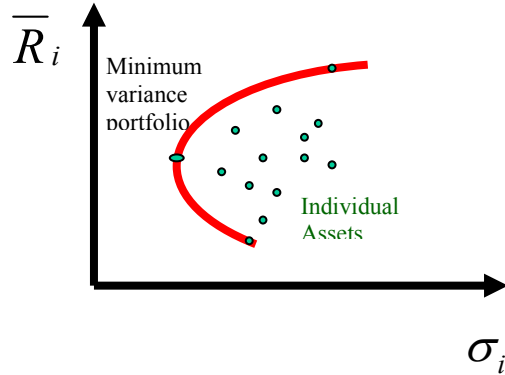
The feasible set includes all available risky stocks and portfolios made up of these stocks. They plot on the curve or inside the curve. The minimum variance portfolio is the portfolio which has the least risk in the feasible set.

The Efficient Frontier and Indifference Curves

We now want to focus on which of these assets or portfolios is desirable to the investor. In fact, it might be good to consider which would be his favorite investment. First, we assume that investors like expected return but dislike risk. Investors are willing to accept risky assets, but they want to be compensated for it by increased expected return. Based just on this, it is clear that some portfolios dominate others. For example, any portfolio which provides greater expected return at the same risk or less risk at the same expected return dominates. The portfolios that dominate make up the efficient frontier.
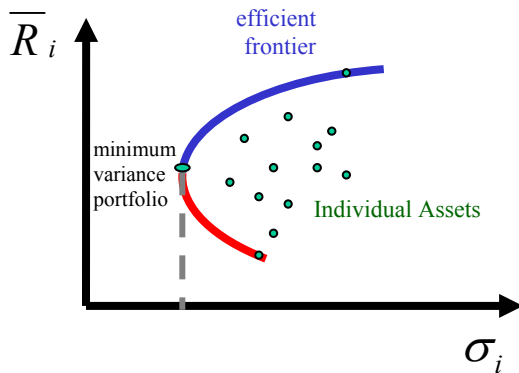

The curve **above** the minimum variance portfolio is efficient. All stocks and portfolios below the efficient frontier are considered inefficient because another stock or portfolio dominates in terms of having either less risk with equal expected return or greater expected return at the same risk. The individual assets in the graph above are considered inefficient as well as the curve below the minimum variance portfolio.
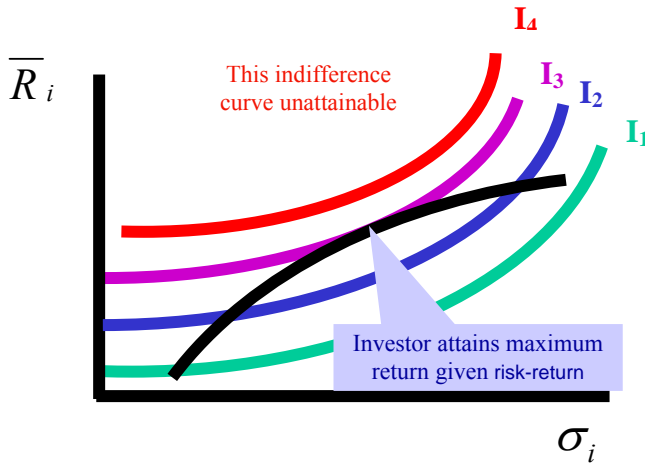
The next issue would be to examine how investors differ from each other, and how that would affect their portfolio selection. Although we assume that investors are risk averse, they have different trade-offs regarding expected return and standard deviation of returns. The tool we use to capture these differences is called the indifference curve. This indicates an individual's expected return – standard deviation of return (risk) trade-off. An indifference curve is a curve in expected return – standard deviation space, where we are equally satisfied as we move along the curve. We are equally satisfied on the same curve but could be better off on a different indifference curve.

What you need to imagine is that there exists, for a particular investor, an infinite number of these curves, and that they are increasingly better for the investor as he moves to a higher curve in the northwest direction (up the graph and toward the left). The investor will have increased expected return or less risk.   What makes one set of indifference curves different from another? The basic shape which defines the risk-return trade-off!

Observe the graphs below, which shows indifference curves for two different investors. For an identical increase in risk (quantity x), indifference curve B requires more expected return than indifference curve A. Indifference curve B represents greater risk aversion (dislike of risk).
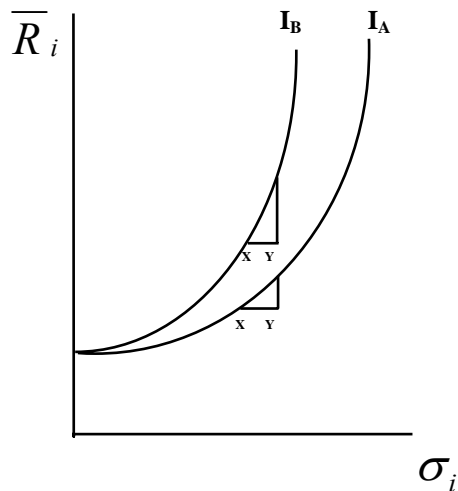
The graph below shows the efficient frontier, and four indifference curves of a single investor:

$\overline{R}_i$

I₄

This indifference
curve unattainable

I₃ **I₂**

I₁

Investor attains maximum
return given risk-return

σ_i

The investor would prefer I_4 but the highest point on the feasible set (it happens to be on the efficient frontier) can only reach I_3. His optimal point is the tangency between his highest indifference curve and the efficient frontier. Not everyone will pick the same optimal portfolio. Another investor may have less risk aversion and will select a more risky portfolio.

In the diagram below, investor x has a different optimal portfolio on the efficient frontier compared to investor y, who is less risk averse.

$\overline{R}_i$

investor y

investor x

Efficient
frontier

σ_i
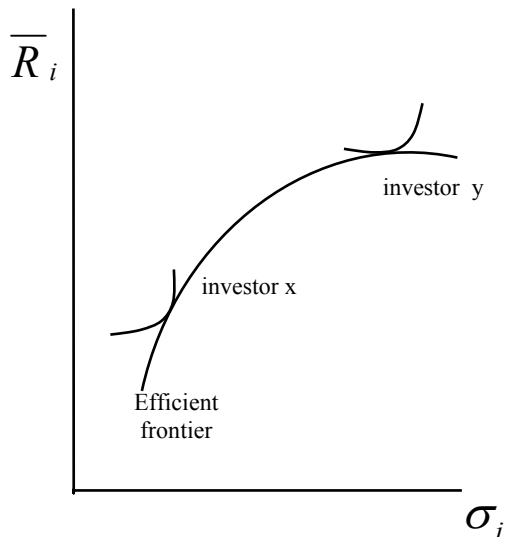
The Risk-Free Asset and the Market Portfolio

As you can see, when we have a set of risky assets but no risk-free asset, different investors will have different optimal portfolios, but all of these will be on the efficient frontier. What if we introduce a risk-free asset? The covariance between a risk-free asset and a risky asset is 0, but the standard deviation of the risk-free asset is also 0. It can be shown that the portfolio possibility between a risk-free asset and a risky one will also be on a straight line between the two assets in the expected return – standard deviation space, similar to the two asset case with $\rho = 1$ above.
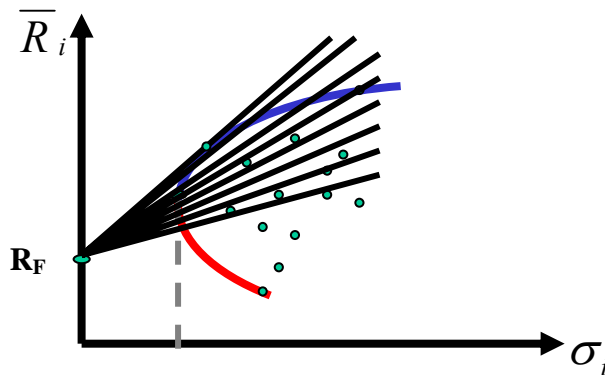
The above diagram shows the possibilities of combining the risk-free asset with any stock in the feasible set. But one line dominates above all the others. The one which is tangent to the efficient frontier! But since all investors want to be on this line regardless of their risk-aversion, this becomes the new efficient frontier. The point of tangency is the portfolio which everyone will want to hold, so we call it the market portfolio.
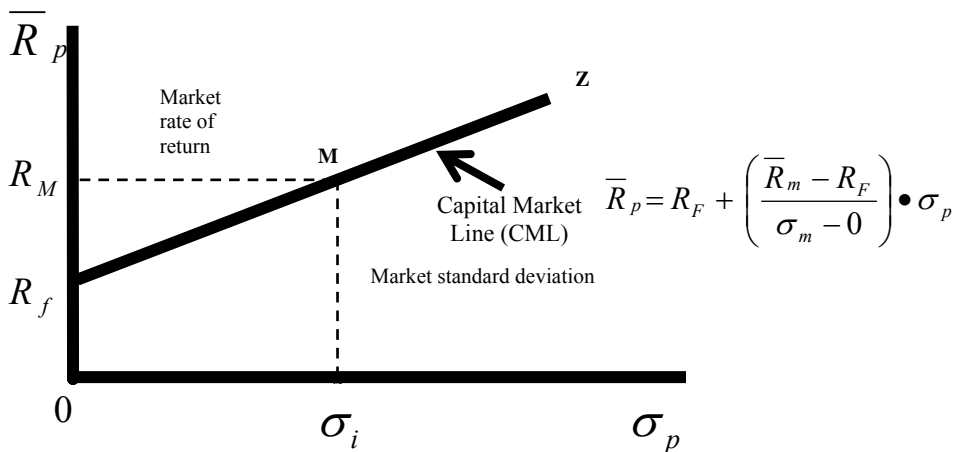
In this theory, everyone agrees to the expected return and standard deviation of returns of all the individual assets, so everyone is looking at the same feasible set of assets. After introducing a risk-free asset, everyone is holding the market portfolio since it dominates as the risky asset to mix with the risk-free asset. The market portfolio contains all the assets that the investors decide to hold collectively. This is why theoretically, the market portfolio contains "all the stocks that anyone holds", and this is why we call it the market portfolio.

Although everyone holds the market portfolio in combination with the risk-free asset, not everyone is holding the same mixture of these two assets. A more risk averse investor may invest more in the risk-free asset while a less risk averse investor will invest more in the risky market portfolio. An even less risk averse investor might borrow money and be more than 100% invested in the risky assets.
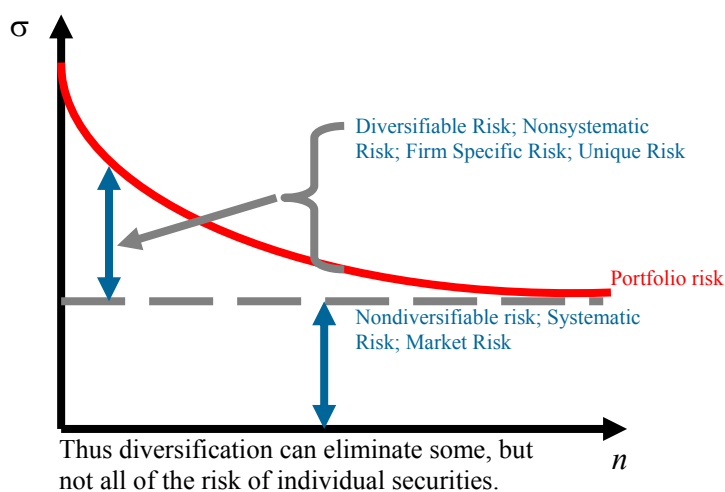
$\overline{R}_i$

P   Z

M

O

R_F

σ_i
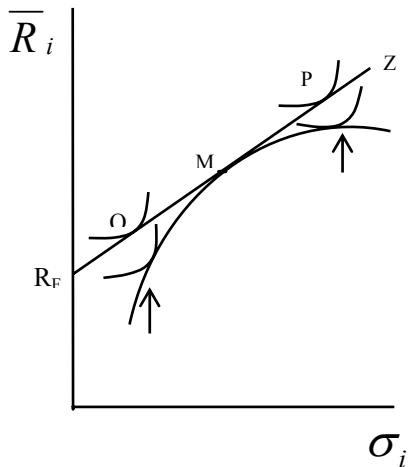
Capital Market Line

In the diagram above, investor O is more risk averse, and has some funds in the risk-free asset and some in the market portfolio M, while investor P is less risk averse and has actually borrowed money at the risk-free rate and moved to the right of point M on the line R_FZ. This line is also called the Capital Market Line (CML).

$\overline{R}_p$

Z

Market
rate of
return

M

R_M

Capital Market
Line (CML)

$$\overline{R}_p = R_F + \left(\frac{\overline{R}_m - R_F}{\sigma_m - 0} \right) \bullet \sigma_p$$

Market standard deviation

R_f

0

σ_i σ_p

The formula of the capital market line is easy to figure out from the diagram. One important point: It holds only for the portfolios on the CML, which are the efficient portfolios and which are also a mixture of the market portfolio and the risk-free rate of return.   The quantity in the brackets is the slope of the equation. The numerator of the slope is the risk premium of the market portfolio, while the denominator is the standard deviation of the market portfolio. One interpretation of this quantity is the "price of risk". For a given portfolio which uses only the market portfolio as a risky asset, we take the risk that we want to accept, multiply by the "price of the risk" and we obtain the risk premium for our portfolio, which we then add to the risk-free rate. But don't forget this only holds for efficient portfolios using the market portfolio.

If we start with some individual stocks and add other stocks to form a diversifiable portfolio, as we add our stocks, we start to reduce our risk. In theory, we can diversify away all risk accept the "market risk". What this means is that as we move toward holding only the market portfolio, we reduce all the risk that can be diversified away. What is left is the risk of the market portfolio. As you can see from the graph below, there are a number of terms which mean the same identical concept.

Diversifiable Risk; Nonsystematic Risk; Firm Specific Risk; Unique Risk

Portfolio risk

Nondiversifiable risk; Systematic Risk; Market Risk

Thus diversification can eliminate some, but not all of the risk of individual securities.

The Capital Asset Pricing Model

The final result we want to cover is a theory which was discovered shortly after the development of portfolio theory. The question it attempts to answer is as follows: If everyone follows the previous suggestions about holding the market portfolio as the risky asset, can we determine what should be the expected return of an individual stock? The result is the capital asset pricing model (CAPM). Although the name says "pricing model" it is actually a model for "expected return" but there is a close connection between a price and a return.

When we write out formula for the variance of a portfolio, we see that it is made up of the proportions we invest in each asset, and the variances and the covariances of the individual assets. It is not evident from the formula for two assets, but for n assets, the covariances outnumber the variances. For an n-asset portfolio, there will be n variance terms but $n^2 - n$ covariance terms. The covariance terms are more important in determining the risk of the portfolio. It can be shown that as we move toward a well diversified portfolio like the market portfolio, the marginal contribution to the risk of the portfolio by the individual asset is a concept known as the beta of the stock. Beta is primarily determined by the covariance of the stock with the market portfolio.

$$\beta_i = \frac{Cov(R_i, R_m)}{\sigma_m^2}$$

The variance of the market return (the denominator above) will be the same for all stocks, so the important variable in beta is the covariance term between the individual asset and the market portfolio. Both this fact and the example of the n asset portfolio variance calculation dominated by covariance terms, give support (though not actual proof) that the relevant measure of risk of an individual asset, when we hold it in a portfolio, is its beta and not its standard deviation.

It is also helpful to look at a plot of the returns to an asset on the vertical axis and the returns to the market portfolio on the horizontal axis. If we fit the best line to these points (regression analysis) the slope of the line is equal to beta, which is the covariance between the asset's return and the return on the market, divided by the variance of the market return. This is regression line is also known as the characteristic line. It is one way to estimate the beta of a stock.

In the diagram below, we have an example of a characteristic line. This is not an expected return – standard deviation of return plot of assets, but a plot of a pair of returns, on the y axis, the stock return and on the x axis the return on a market portfolio. We fit the best straight line to the points using simple linear regression, and the slope of this line is the beta of the stock.

Stock return R_i (%)

$$R_i = a_i + \beta_i R_M + e_i$$
$$.42 + 1.20\ R_M$$

Market return R_M (%)
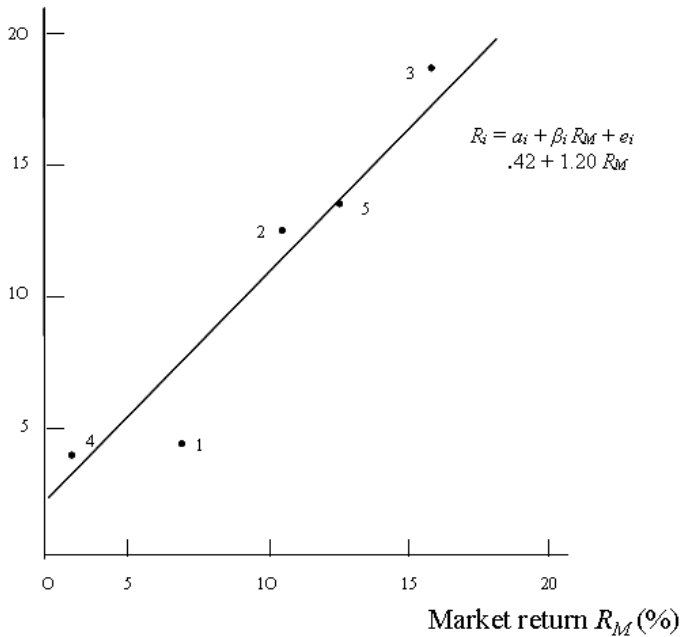
By holding assets in portfolios, we can eliminate some of the risk of the individual assets. This is the result of portfolio diversification. The part we eliminate is called the unsystematic risk. The rest of the risk, which is captured by beta, is called the systematic risk. It can not be diversified away. If we assume that all investors hold portfolios to take advantage of their diversification benefits, they price of the individual assets that make up the portfolio will reflect expected return to compensate investors only for the systematic risk, or the risk captured by beta. This is the principle behind the capital asset pricing model.

We illustrate this with another graph. The vertical axis is the same expected return as we had for the CML, but the horizontal axis will now be beta. The equation is a straight line that passes through the return of the risk-free asset. The slope of the line will be the expected return of the market portfolio minus the risk-free rate. This is known as the security market line (SML). Unlike the CML, this equation holds for all assets.

$$\overline{R_i} = R_f + \beta_i(\overline{R_m} - R_f)$$
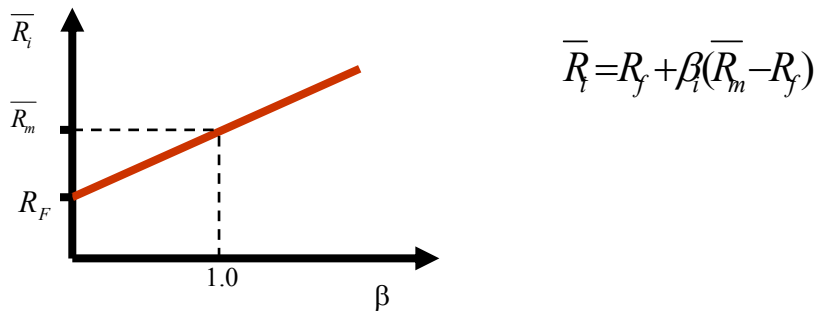
Knowing what beta is (covariance) and properties of covariance (covariance of a variable with itself is the variance; covariance of variable with a constant is zero) we can easily obtain results such as beta of the market portfolio is 1,and the beta of the risk-free asset is zero. It is then an easy step to write down the equation of the security market line.

Suppose that the risk-free rate is 3%, the expected return on the market is 10% and the beta of the stock is 1.5. The Capital Asset Pricing Model will indicate that the expected return on the stock should be 13.5%.

$$\overline{R_i} = 3 + 1.5(10 - 3) = 13.5\%$$
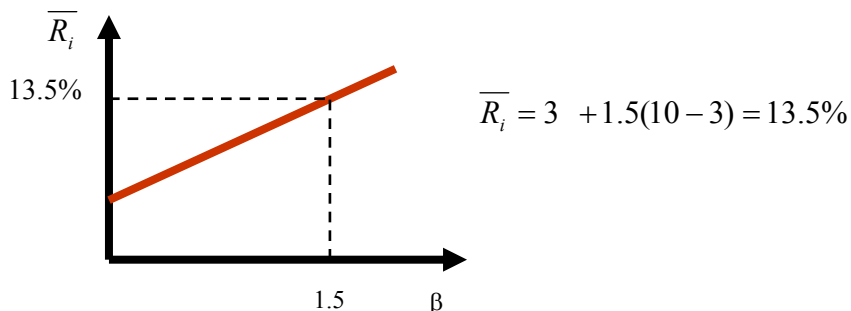
It is useful to draw the two graphs with the common vertical axis side by side. I handed out such a diagram in class. The common vertical axis stands for the expected return of an asset or a portfolio. Let the graph on the left-hand side have a horizontal axis showing beta, therefore it can be used to plot the SML. The right hand side will have a horizontal axis that shows standard deviation of returns; therefore it can be used to plot the CML. The diagram below just shows the same two diagrams you have seen above. It is as if we are examining three properties of returns at the same time (sort of like a 3 D graph): The expected return, the standard deviation of returns, and the the beta of the securities.
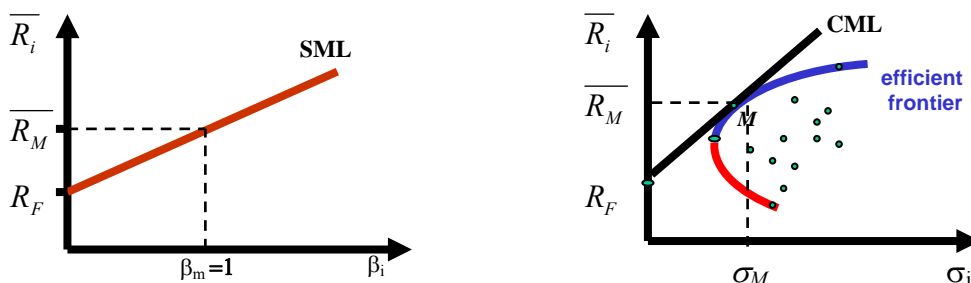
There are some differences between the two graphs. There are many single stocks and portfolios of stocks below the efficient frontier in the CML diagram on the right hand side. They are all inefficient after the risk-free rate is introduced because the new efficient frontier is the capital market line (CML) which is the ray from the risk-free rate going through M, the market portfolio. Only the portfolios on this line are efficient.

But in the left hand security market line (SML) diagram, you only find assets plotting on ray from the risk-free rate. When the market is in equilibrium, all assets will have the expected return as predicted by the capital asset pricing model (CAPM), and its equation is actually the SML. Although there are many stocks and portfolios in the right hand diagram, they all plot on the left hand diagram on the same line. Where on the line? Exactly at the same height in both diagrams, because the y axis is identical for both diagrams! The equilibrium position as determined by the CAPM will determine the expected return on the stock.

Below, you will see the same two diagrams with three specific stocks or portfolios indicated. Stock A is in the interior of the efficient frontier. Portfolio B is an efficient portfolio made up of the risk-free asset and the market portfolio, which provides the same expected return as stock A. Portfolio C is also an efficient portfolio made up of the risk-free rate and the market portfolio (in this case it is obtained by borrowing at the risk-free rate and levering the market portfolio). Portfolio C has identical standard deviation to stock A.
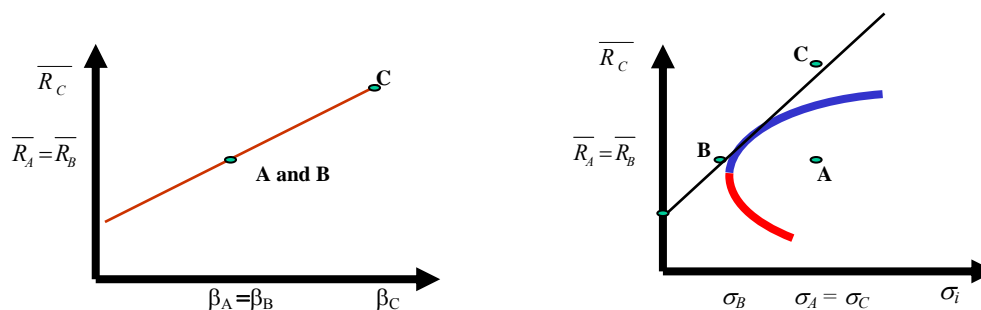
As you can see, portfolio B is on the same level as stock A. It has the same expected return as A but with less standard deviation. This portfolio, because it is on the CML will be "an efficient portfolio". Both stock A and portfolio B has identical beta and it is this beta that actually determines the expected return of both!   The efficient portfolio has only systematic risk (beta risk) since the unsystematic risk has been diversified away. (See the diagram on page 10.)

The expected return is the fair return for the systematic risk. If you hold asset A alone, you are exposed to more risk (the unsystematic risk) but you will not be compensated for holding this risk with more expected return. This is why you are at a disadvantage if you do not diversify away the unsystematic risk by plunging in a single stock. Notice that asset C, which plots on the Capital Market Line, has the same standard deviation as asset A, but its expected return is much higher. It is obtained as a combination of the market portfolio M and the risk-free asset R_F. In this case, you would short the risk-free asset (borrow) and leverage the market portfolio for a higher return. Asset C is efficient because it is on the CML. All the diversifiable risk (unsystematic) has been eliminated by holding the market portfolio. The beta of asset C determines its expected return, and it is higher than asset A. Although portfolio C is more risky than asset B, it is compensated by the higher expected return. Both B and C are efficient portfolios, and their beta determines their expected return, but asset A is inefficient and has diversifiable risk in it, which is not compensated by its expected return. The market uses only efficient portfolios to set the fair price which determines their expected return.

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