Lecture Notes 8: The Forward Contract

Introduction and Pedagogical Plan

Our coverage of the forward contract (forward, for short) will parallel the exposition we used for options. We start with the definition of forwards as well as other terms used for describing and valuing this instrument. This is followed by the notation we can use for equations. Very similar to options, equations allow for exact determination of values and they can be used to graph value and profit diagrams. As before, the argument is that diagrams helps to visualize the potential costs and benefits of using a particular security.

It may seem that by devoting only a few weeks to forwards and futures, and only a single week for swaps, we are short-changing the study of these assets in favor of options. But consider the fact that you needed to learn both the creation of profit diagrams as well as the concept of arbitrage while you learned about the institutional detail of options. Now that you know both profit diagrams and arbitrage, you only need to understand the difference between forwards, futures, and swaps, and the calls and puts you are already familiar with.

Definitions

Forward Contract  An agreement to buy or sell an asset on a date in the future for a price specified today. Often we will simply call it a forward.

Forward Price  The specified selling price in the forward contract. It plays a similar role that the strike price plays in option contracts.

Expiration Date  The date delivery is made of the asset in return for the forward price. Also called delivery or settlement day.

Spot Price  This is the price of an asset for immediate delivery, or the current price of the asset. It is sometimes called the cash price.

Futures Contract  A futures contract is a forward contract trading on an organized exchange. Forward contracts are standardized, and futures markets have a mechanism in place which minimizes exposure to the default of the counterparties. This mechanism is the daily marking-to-market of futures contracts.

A futures contract is similar to a forward contract, but it trades in an organized exchange. The similarity to options is as follows: A forward contract trades in a manner similar to over-the-counter options while a future contract trades in a manner similar to options on the organized options exchanges. This learning module will be focused on the forward contract, while the next one will focus on futures.

What is the essential difference between a forward and an option?
The essential difference is that with a forward contract, you no longer have an option to exercise! You must go through with the purchase of the underlying asset (if long in the forward, meaning you were the buyer of the forward) or the sale of the underlying (if short in the forward, meaning you sold the forward).

Notice that for each forward there is a long and a short position, but this takes care of all possibilities regarding the future purchase and sale of the underlying asset; there is no need for a “forward call” or a “forward put”. But this is not really an advantage over options. In some ways, options are an even more basic building block than a forward contract or buying the underlying stock directly.

For a forward contract:

1. Buyer has an obligation just like the seller
2. The position of the buyer and seller is much more symmetrical than with options
3. There is no reason to have a premium
4. Still, you need a mechanism to bring equilibrium between buyers and sellers. The forward price is allowed to fluctuate and is the equilibrating factor. At the “correct” forward price, there should be an equal number of buyers and sellers in agreement about the “fair” forward price, just as in the options markets the option premium acts as the equilibrating factor.

Perhaps these points need a bit more explanation. With options, a writer can never benefit from the position after the option is sold. The buyer will only exercise if it is beneficial for him or herself. It is for this reason that a premium is paid from the option buyer to the writer. The amount of the premium is determined by the same variables that you used for the Black-Scholes formula, and one of these is the strike price.

Although the theoretical formula determines the option value, the market price is actually determined by the buyers and sellers of that option. That is, buyers and sellers determine the actual market price, while the Black-Scholes formula is just a model for the market price. It is the premium which moves up and down in response to the actions of traders.

To summarize for options: For a given strike price, which is often selected by the options exchange, buyers and sellers are brought into equilibrium by the fluctuating market price (premium) for the option.

Now for a forward contract, the forward price acts in the same capacity as the strike price for options. It is the agreed upon price for the underlying asset. But for a forward, there is no premium paid by the buyer to the seller. The payoff for the either side is not lopsided as it is for options, where the writer will never benefit in the future. In fact, the deal takes place when forward buyers and sellers are equally comfortable with the proposed forward price. The job of bringing the market into equilibrium is done with the fluctuating forward price.

**Notation**
Let us use the following notation:

- $S_t$: spot price at time $t$
- $F_t$: forward price at time $t$
- $V_t$: value of a forward contract at time $t$
- $T$: time to expiration and also expiration date if contract was initiated at $t = 0$ (see notes related to this point in Lecture Notes 1 on options)

Also we can modify above to indicate both the contract initiation date and expiration date by using the following:

- $F_t(0,T)$: This is the forward price at time $t$ (a specific time), initiated at $t = 0$, and delivery at $t = T$.
- $V_t(0,T)$: This is the value of a forward at time $t$ (a specific time), initiated at $t = 0$, and delivery at $t = T$.

We will attempt to use the same notation as used in Chance and Brooks (CB), but will also simplify whenever possible. For example, CB always wrote call values as $C(S_0, T, X)$ while we just wrote $C_0$ or $C_t$ if that was all that was required. If it is clear that we are initiating at $t = 0$ and delivering at $t = T$, we will just write the forward price as $F_0$ if we want to discuss the forward price at the initiation of the contract, or $F_t$ if we mean the forward price sometime later. If it is clear which forward price we mean, we may even just write $F$.

But sometimes we need to be more explicit. We should always have $T$ in there if we are at expiration, and often we will talk about a contract initiated, and then additional contracts initiated. Since the second contract will most likely have a different forward price, we must then be quite explicit with our notation.

**Value of a Forward at Expiration**

**Example:** Suppose the spot price of gold is $400/oz. We will drop the $ and oz notation for simplification. Suppose I purchase a forward contract to buy an ounce of gold one year from today at a forward price of 460. I am long one forward (one ounce per contract) for settlement/delivery one year into the future.

- $S_0 = 400$
- $F_0 = 460$

$V_0 = 0$  

Why?
Both buyer and seller are equally satisfied regarding forward price of 460, otherwise they would not make the deal. They shake hands and no money needs to change hands, since there is no premium for forwards. It makes sense to say that the value of this contract the moment it is made is essentially zero. But almost immediately the value of this contract could change since the forward price of newer contracts for the same expiration date could be quite different.

What is the value of this forward contract at expiration? Is it a different value to the buyer and the seller? Yes!

This is not that hard to determine, same as with options:

\[ V_T = S_T - F \] to the buyer

\[ V_T = F - S_T \] to the seller

Notice that there is no need to have max function here. Why?

But of course, this value depends totally on the value of the spot price at expiration!

Suppose that \( S_T = 500 \)

Value to the buyer = \( 500 - 460 = 40 \)

Value to the seller = - 40

We could now ask a few questions:

Is this a zero-sum game? Yes, to some degree. This means that what the buyer will wins the seller will lose, but there is more to this than just that.

Was the buyer a hedger or a speculator?

Are forward contracts risky?

Are there social benefits of forward contracts?

What is the role of a “speculator”? Does he benefit society?

Before we can answer some of these questions, we need a few definitions.

**Hedger**

One who has some risk and manages to transfer some or all of it to another party.

**Speculator**

One who takes on risk for expected profit. This sounds a bit like an “investor”.

Financial Derivatives ©Steven Freund
Short Hedger  
A hedger who is selling a forward contract (or similar derivative).  
Example:  Gold mine owner hedging the risk of fluctuating gold prices.

Long Hedger  
A hedger buying the forward.  Example: A Manufacturer that needs gold in his production.  He is also hedging price risk.

Suppose you need gold in the future.  Perhaps you are the manufacture above that needs gold in your production.  You are exposed to the risk of fluctuation in the gold price.  But if you buy a gold forward, you have now hedged the price risk.

And if you are a gold mine owner, you can sell your future production through the use of a forward contract.  You have also hedged your price risk.  These two examples show that two hedgers can make a deal, which lowers risk for both!  We can answer the previous questions now.

Was the buyer a hedger or a speculator?  It can’t be determined unless we know why he or she entered the market.

Are forward contracts risky?  Yes, they are, but as you can see they can also lower the risk for hedgers.  If you are a speculator and buy or sell a forward contract, you have increased your risk.  These would be people who have no need to buy or sell the underlying asset in the future.  In fact, the risk of forwards is even more risky than just buying the underlying asset due to greater leverage.

Are there social benefits of forward contracts?  It can help transfer risk from people who do not want to hold it.  And as you could see with the above example, two hedgers can make risk totally go away.

What is the role of a “speculator”?  Although he is in it to make money, that is increase expected return in exchange for taking on risk, he can also help with the transfer of risk from those who want to get rid of it.

Does he benefit society?  Speculators can make for a better market for the hedgers.  They take care of possible imbalances between short hedgers and long hedgers.

The Cost of Carry Model for Forwards

If you are a long hedger there is another way to hedge.  You could just buy the spot asset outright at \( t = 0 \).  But if you do, you will have carrying costs:

1. Storage
2. Insurance
3. Interest Expense (opportunity cost of money)
Let $\theta$ denote the above costs. The cost-of-carry model is:

$$F_0 = S_0 + \theta$$

How can we show this? We will use arbitrage. Suppose

$F_0 > S_0 + \theta$?

Is there a strategy for someone who has no interest in gold but likes arbitrage profits?

Gold Example: Suppose $F_0 = 480$ when $S_0 = 400$, interest is 10% per year, and storage and insurance for one ounce of gold for a year is 20, paid in advance!

You need to borrow enough to pay 420 today. At expiration you will need to pay back $420(1 + .10)$ or 462. Your interest expense at $t = T$ will be 42 for financing both the storage and the purchase of the gold.

In this example, $\theta = 62$

So here, $F_0 = 480$, while $S_0 + \theta = 400 + 62 = 462$

Clearly $F_0 > S_0 + \theta$ in this example.

The arbitrage should be as follows:

Borrow enough money to buy one ounce of gold and store it for a year, and also sell a forward contract for one ounce.

<table>
<thead>
<tr>
<th>$t = 0$</th>
<th>$t = T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>buy gold</td>
<td>- 400</td>
</tr>
<tr>
<td>pay storage</td>
<td>- 20</td>
</tr>
<tr>
<td>borrow</td>
<td>+ 420</td>
</tr>
<tr>
<td>sell forward</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>Arbitrage Profit</td>
<td>0</td>
</tr>
</tbody>
</table>

Some Comments:

1. This is similar to the arbitrage we did in options, except
2. Positive cash flow at $t = 1$, zero at $t = 0$, but this is still arbitrage.
3. Notice zero cash flow when you sell the forward
Suppose $F_0 < S_0 + \theta$

For example, $F = 440 < 400 + 62 = S_0 + \theta$.

Clearly you should buy the forward, but now you need to sell gold. You can do this if you own some, and you will save storage and earn interest on the selling price and storage cost.

<table>
<thead>
<tr>
<th>Time</th>
<th>Action</th>
<th>Amount</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t=0$</td>
<td>sell 1 oz gold</td>
<td>+ 400</td>
</tr>
<tr>
<td></td>
<td>save storage</td>
<td>+ 20</td>
</tr>
<tr>
<td></td>
<td>lend</td>
<td>− 420</td>
</tr>
<tr>
<td></td>
<td>receive</td>
<td>420*(1.1) = + 462</td>
</tr>
<tr>
<td>$t=T$</td>
<td>buy forward</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>pay $F$</td>
<td>− 440</td>
</tr>
<tr>
<td></td>
<td>and take delivery</td>
<td></td>
</tr>
</tbody>
</table>

Arbitrage Profit: 0 $\rightarrow$ 22

When you take delivery of the gold at $t=T$, you get your gold back, as well as $22 arbitrage profits.

We can put these two results together (a bit like put-call parity) and we have demonstrated the cost-of-carry model:

$$F_t = S_t + \theta$$

where $F_t$ and $S_t$ are the contemporaneous forward and spot prices on the same commodity. As you can see, the cost-of-carry is a relation between the spot price of a commodity and its forward price at the same point in time. $\theta$ is the cost to carry the spot commodity forward from time $t$ to time $T$, the settlement of the forward contract.

We have demonstrated that if it doesn’t hold, there will be an arbitrage opportunity. If $\theta > 0$ (same as $F > S$) the cost-of-carry is positive. There are a lot of different terms related to this relationship in the industry, and it is quite confusing. Some terms you may see include carry, carry market, negative or positive carry.

Of course, in the cost-of-carry model, $\theta$ is really the net cost-of-carry because there could be positive benefits to holding the cash asset (another name for the spot or underlying asset) rather than the forward. For example, if the underlying asset is stock, and it pays a dividend, then dividends are a positive benefit from holding the cash asset. You would subtract out the dividends from the costs of holding the cash asset over the forward. This is why we say $\theta$ is actually a net cost-of-carry.

To summarize the cost of carry model is:

$$F_t = S_t + \theta$$
If $F_t > S_t + \theta$, even if we have no interest in the spot commodity, we could buy it, and simultaneously establish a short position in the forward.

If $F_t < S_t + \theta$, we need to sell the spot commodity, which not everyone can do. Still, we argued that enough people could do it such that the cost-of-carry model should still hold. When you temporarily sell the spot asset you own to perform an arbitrage, we call it quasi arbitrage.

Of course, the commodity must be a storable item for this model to hold.

**The Basis**

We define “basis” as:

$$b_t = S_t - F_t$$

In words, the basis is the difference between the spot price and the forward price. The basis is negative when the cost-of-carry is positive. Notice that if the cost-of-carry is positive, the contemporaneous spot price will be below the forward price, and consequently, the basis will be negative. Also, the forward will then plot above the spot price.

This above situation shows a **negative basis market**. This situation is also called **contango**.
Also notice as we move toward expiration, the forward price will converge to the spot price. The cost-of-carry becomes less as we move toward expiration, which should make sense!

It is possible that your cost-of-carry term is negative. For example, dividends on the underlying asset could outweigh both your interest cost and your storage cost. In that case, the cost-of-carry is actually negative, and the basis becomes positive.

Sometimes an added benefit will be called a convenience yield. This also will decrease the effective cost-of-carry. An example of a convenience yield is something which yields a convenience for carrying the cash asset. For example, if you were hedging a need for future gold, which you need a year from now, the fact that you have the gold already means that if there is a chance you need gold earlier, you have it in stock.

If the effective or net cost-of-carry is negative, the basis is positive, and the forward price would be below the current spot price. The diagram on the previous page would be reversed with respect to the position of the forward and spot prices. This situation is called backwardation or an inverted market.

The previous situation of a negative basis or contango is more common.

**Forward Value before Settlement**

In the above sections, we have shown how to value a forward at expiration, and also at the time we initiate the contract \( V_0 = 0 \). In this section we will show how to value the forward at a point between \( t = 0 \) and \( t = T \).

We need to discuss notation a bit. Recall that sometimes all we need to indicate is where we are in time. For example, the spot price \( S_t \) needs the subscript \( t \) to indicate where we are: for example, \( S_0 \) or \( S_T \). This is the spot commodity at time \( t = 0 \) or at time \( t = T \).

Now consider a forward price \( F \). Previously, we initiated all contracts at \( t = 0 \), so we could just write \( F \). Then we added a subscript to show that the forward price varies as we vary the point where we are relative to a given expiration. If the expiration is always at \( t = T \), we can ignore it and just focus on where we are in time. So we wrote \( F_t \). Go back and see diagram at the bottom of the previous page. If we want to make the expiration point explicit, we could use the following notation:

\[
F(t,T)
\]

which now indicates the forward price at time \( t \) for a forward contract with delivery at time \( T \). In fact, this would be identical to \( F_t \) but the expiration then is not indicated specifically.

Numerical example: Suppose \( t = 10 \). Let us assume we are using units of months, so expiration is 10 months from today. Suppose that today the forward price is 400, but next
month, for the same expiration which now is only nine months away, the forward price is now 405.

We could indicate this as $F(0,10) = 400$, and $F(1,10) = 405$. More simply we could just write $F_0 = 400$, and $F_1 = 405$, but then we no longer can tell when expiration will be, or in fact that both of these forward prices are for the same expiration.

Once the forward price is set, then for that given contract, that will always be the forward price. But the value of that forward contract can change as we move through time toward expiration. The notation for the value of the forward really needs three points of time to totally define which contract we are discussing.

Suppose we have a situation where we initiate a forward contract at time $t = 0$ for settlement at $t = T$, but in between, say at $t = t'$, we want to figure out the new value of the original forward. In general, we would want to do this for any $t = t'$ between 0 and $T$, and what we want is:

$$V_{t'}(t,T)$$

In words, we have a forward contract with expiration $t = T$, that was initiated at time $t$, with a forward price $F(t,T)$, and we want to value it at time $t = t'$.

The notation required here is a bit awkward, but if you can get through the next section, and then consider an actual example with numerical numbers, I think it will become clear. Let’s use a time diagram to help us:

\[
\begin{align*}
\text{t = 0} & \quad \text{t = t'} & \quad \text{t = T} \\
S_0 & \quad S_{t'} & \quad S_T \\
F(0,T) & \quad F(t',T) & \quad F(T,T) = S_T \\
V_{0}(0,T) = 0 & \quad V_{t'}(0,T) = ? & \quad V_{T}(0,T) = S_T - F(0,T) \quad \text{(to the buyer!)} \\
V_{t'}(t',T) = 0 & \quad V_{T}(0,T) = S_T - F(t',T) \quad \text{(to the buyer!)}
\end{align*}
\]

So our objective here is to figure out the unknown value of the forward contract initiated at $t = 0$ and expiring at $t = T$ at the point where $t = t'$. Everywhere else for all the other variables everything is already known by us (or at least should be!).

Let us just go over the previous diagram. The first line below the time line indicates where we are. The second line indicates the spot price at these points. The third line
indicates the forward prices determined at these points in time. Notice that a forward price for a forward contract initiated at time T and with expiration at time T is identical to the spot commodity. The third line follows the value of a forward initiated at t = 0, as we move through time toward expiration. We know the value at t = 0, and t = T, but not yet at t = t’. The last line is related to the value of a forward contract initiated at t = t’.

I think by studying this a bit, you will get real familiar with this notation. Now let us consider how to value the forward initiated at t = 0 but at t = t’. We will actually use the other forward at t = t’, and its forward price to value the original forward.

Consider a portfolio consisting of a long forward established at t = 0 and a short forward established at t = t’.

This portfolio will have the following value at t = T:

\[ V_T(0,T) - V_T(t',T) = [S_T - F(0,T)] - [S_T - F(t',T)] \]

\[ = F(t',T) - F(0,T) \] (the two S_T s cancel)

Notice that the risky element in this portfolio, the spot price at t = T has dropped out. We could now figure out the value of this portfolio at t = t’ but discounting back from t = T:

\[ V_{t'}(0,T) - V_{t'}(t',T) = [F(t',T) - F(0,T)](1 + r)^{-(T - t')} \]

and since \( V_{t'}(t',T) = 0 \)

\[ V_{t'}(0,T) = [F(t',T) - F(0,T)](1 + r)^{-(T - t')} \]

We could also explain all this in words, but let us try an example first:

\( S_0 = 400 \quad F_0 = 460 \quad T = 1 \text{ year} \quad r = .10 \)

At \( t = .5, \) if \( F_{.5} = 480 \)

\[ V_{.5}(0,1) = (480 - 460)(1.1)^{-5} = 19.06925 \]

You buy a forward contract at t = 0, with a forward price of 460. Six months later, forwards for the same expiration are selling for 480. To lock in your profit at this point in time, sell a forward for 480. Essentially you are doing an offset. But since these are forwards, you need to wait until the expiration of both your long and your short to obtain your profit. If you discount $20 for the remaining 6 months at ten percent interest, your forward value at t = .5 will be $19.07.
Relation between the Forward Price and the Expected Spot Price

Previously we have considered the Cost-of-Carry model for the forward. It actually describes the contemporaneous relationship between the forward price and the spot price. This is another important relation in the forward/futures literature. It is the relation between the forward price today and the expected spot price at the expiration of the forward contract.

Let us review the meaning of expected value. If we have a random variable \( x \), whose outcome can only be described by a probability distribution, we can summarize the distribution by such measures as population mean and the population standard deviation. The mean is the expected value of the random variable, denoted by \( E(x) \). We can estimate its value by the sample mean or average, and we define risk as the standard deviation of the distribution.

We are going to argue below that only if speculators are risk neutral, which means that they do not need to be compensated for carrying risk, are they willing to have the forward price be equal to the expected spot price.

That is, if \( E(S_T) = F \), on the average they will not make any money. If they buy the forward, they will make money if \( S_T > F \), and lose money if \( S_T < F \). But on the average, they will get zero return. Let’s show that: Suppose you are a speculator that has gone long in the forward. If the expected spot price at expiration equals the forward price, \( E(S_T) = F \), then the expected value at expiration will be zero.

\[
E(S_T - F) = E(S_T) - E(F) = F - F = 0
\]

The reason \( E(F) = F \) is because \( F \) is a constant, not a random variable; the zero result for the payoff to the long position might be ok, because you did not have a cash flow at \( t=0 \). But there is risk, because this is zero only on the average. But if you are a speculator, you want a greater reward for holding risk! But in finance, we usually assume that investors are risk averse, and will accept risk only if compensated with higher expected return. Since speculators always play a role in these markets, and they are not likely to be risk neutral, the equation: \( F = E(S_T) \) is not likely to hold.

The following was first developed by John Maynard Keynes, and it uses the idea that there are two types of participants in a forward or futures market: Speculators and Hedgers.

Hedgers may need to go long or short, but speculators will do either if it is worthwhile for them to do it. If \( F = E(S_T) \), they will probably not enter the market since they are taking a risk without being compensated for it. If \( F < E(S_T) \), they will want to go long because then:

\[
E(V_T) = E(S_T) - F > 0
\]

The speculators are obtaining a risk premium.
E(S_T) = F + risk premium

Example: If E(S_T) = 500, and you are a speculator you will only go long if F is below 500.

Suppose that there are many more hedgers who want to be short than long; for example more wheat farmers than cereal manufacturers. We say that hedgers are net short. Suppose we only focus on the remaining hedgers who need to go short and have not found a hedger to take care of the long side of the trade. They need to enhance speculators to enter the market. The following diagram will show the supply and demand for these excess long contracts above and beyond the ones where there are hedgers on both sides of the trade. The supply of these contracts is by the net short hedgers. The demand to go long is from the speculators who will only enter the market on the long side if F is below the expected spot price of 500. On the other hand, the higher the forward price, the more contracts the hedgers are willing to sell to the speculators. So the following is the supply – demand for long contracts when hedgers are net short, and need speculators to go long.

When the forward price is below the expected spot price it is called normal backwardation. Notice the additional term “normal” before contango and backwardation. The meaning is different from contango and backwardation without the term “normal”.

When the forward price is above the expected spot price it is called normal contango. This is the situation when hedgers are net long and need the speculators to go short. The forward price then needs to be above the expected spot.