The purpose of this exercise is to develop a proof of the Fundamental Theorem of Calculus using ideas we have developed previously. This is worth up to 10 bonus points added to your homework point total.

Statement of the FTC:

Let $f$ be a continuous function defined on an open interval $I$. Let $a \in I$, and let $A_f(x) = \int_a^x f(t) \, dt$ for $x \in I$. Then $A'_f(x) = f(x)$.

One possible outline of a proof:

1. Show that $A'_f(x) = \lim_{h \to 0} \left[ \frac{1}{h} \int_x^{x+h} f(t) \, dt \right]$.

2. Show that $\frac{1}{h} \int_x^{x+h} f(t) \, dt = f(c)$ for some point $c \in [x, x+h]$.

3. Show that $\lim_{h \to 0} f(c) = f(x)$.

For Step 1, you will need to use the definition of the derivative and a property of definite integrals.

For Step 3, you will need to use the definition of a continuous function.

Step 2 is the trickiest step. Here is one possible path:

1. Let $m$ and $M$ denote the minimum and maximum values, respectively, of $f$ on $[x, x+h]$. Be sure to explain how we know that $m$ and $M$ exist and are finite. (Hint: Look at the Extreme Value Theorem (EVT).)

2. Use the Fact on p. 307 of the textbook to show that $m \leq \frac{1}{h} \int_x^{x+h} f(t) \, dt \leq M$. (Recall that we defined $\bar{f}$ to be the average value of $f$ on $[x, x+h]$.)

3. Use the Intermediate Value Theorem to show that there is some point $c \in [x, x+h]$ for which $f(c) = \bar{f}$. Be sure to explain how you know that $f$ takes on the values $m$ and $M$ somewhere in $[x, x+h]$. (Hint: EVT again.)