Infinite Sets

1. Two sets $A$ and $B$ are said to have the same cardinality if there is a one-to-one, onto function $f : A \to B$.

2. The cardinality of a finite set is the number of elements in the set.

3. A set $A$ is said to be countably infinite (or simply countable) if $A$ has the same cardinality as the set of positive integers $\mathbb{N} = \{1, 2, 3, \ldots\}$.

4. The cardinality of $\mathbb{N}$ is denoted $\aleph_0$. $\aleph_0$ is the first transfinite cardinal.

5. The set of integers $\mathbb{Z} = \{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$ is countable.

Challenge Problem 5. (10 points) Find a one-to-one, onto mapping from $\mathbb{Z}$ to $\mathbb{N}$.

6. The set of rational numbers is countable. (See, for example, Fundamentals of Mathematics, vol. 1, by Behnke et al., MIT Press, 1986, p. 151.)

7. The set $\{x \mid 0 < x < 1\}$ has the same cardinality as the set of all real numbers.

Challenge Problem 6. (10 points) Find a one-to-one, onto mapping from $(0, 1)$ to $(-\infty, \infty)$.

8. The set $\{x \mid 0 < x < 1\}$ is uncountable. The cardinality of $\{x \mid 0 < x < 1\}$ is denoted $\aleph_1$. The proof that $\{x \mid 0 < x < 1\}$ is uncountable is known as Cantor's diagonalization procedure or Cantor's diagonalization argument. The proof is by contradiction. Assume to the contrary that $\{x \mid 0 < x < 1\}$ is countable. Then the numbers in this set can be put into one-to-one correspondence with the positive integers:

\[
x_1 = 0.x_{11}x_{12}x_{13}\ldots \\
x_2 = 0.x_{21}x_{22}x_{23}\ldots \\
x_3 = 0.x_{31}x_{32}x_{33}\ldots \\
\vdots
\]

Now consider the number $a$ whose decimal representation is $a = 0.a_1a_2a_3\ldots$, where $a_i = 1$ if $x_{ii} \neq 1$ and $a_i = 2$ if $x_{ii} = 1$. Then $a \neq x_1$ because $a$ and $x_1$ differ in the first decimal place. Similarly, $a \neq x_2$ because $a$ and $x_2$ differ in the second decimal place. In general, $a$ and $x_i$ differ in the ith decimal place. Thus, $a$ is not on the list of numbers in the set $\{x \mid 0 < x < 1\}$, contradicting our assumption that this set is countable. (See Behnke et al., p. 55.)

9. The Continuum Hypothesis states that there is no transfinite cardinal between $\aleph_0$ and $\aleph_1$. In 1963 Cohen proved that if the axioms of set theory are consistent, then the Continuum Hypothesis is independent of the other axioms. In other words, the Continuum Hypothesis can neither be proved nor disproved from the other axioms. (See, Behnke et al., p. 60.)

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