# Integration Techniques

I. Basic antidifferentiation rules you should know:

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} \quad \text{if } n \neq -1$$

$$\int \frac{1}{x} \, dx = \ln|x|$$

$$\int e^x \, dx = e^x$$

$$\int \sin(x) \, dx = -\cos(x)$$

$$\int \cos(x) \, dx = \sin(x)$$

$$\int \tan(x) \, dx = \ln|\sec(x)| = -\ln|\cos(x)|$$

$$\int \cot(x) \, dx = \ln|\sin(x)| = -\ln|\csc(x)|$$

$$\int \sec^2(x) \, dx = \tan(x)$$

$$\int \csc^2(x) \, dx = -\cot(x)$$

$$\int \sec(x)\tan(x) \, dx = \sec(x)$$

$$\int \csc(x)\cot(x) \, dx = -\csc(x)$$

$$\int \frac{1}{x^2+1}(x) \, dx = \arctan(x)$$

$$\int \frac{1}{\sqrt{1-x^2}} \, dx = \arcsin(x)$$

II. If your integrand does not appear on the above list, it may be possible to simplify the integrand by expanding it.

Example 1. $\int \left(x^2+1\right)^2 \, dx = \int \left[x^4 + 2x^2 + 1\right] \, dx$

Example 2. $\int \frac{x+1}{x^2} \, dx = \int \left[\frac{x}{x^2} + \frac{1}{x^2}\right] \, dx = \int \left[\frac{1}{x} + x^{-2}\right] \, dx$

III. If it is not possible to simplify your integrand, try a substitution. Rules of thumb for deciding what to choose for $u$ when using substitution:

1. If an expression appears raised to a power or under a root, let $u =$ that expression.

   Example. To find $\int \sqrt{2x+5} \, dx$ let $u = 2x+5$.

2. If one part of the integrand is the derivative of a second part of the integrand, let $u =$ the second part.

   Example. To find $\int 3x^2 \sin\left(x^3\right) \, dx$, let $u = x^3$.

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IV. If substitution does not work, try integration by parts: $\int u\;dv = uv - \int v\;du$

Rule of thumb for deciding what to choose for $u$ when using integration by parts: *ILATE* (**I**nverse trig functions, **L**ogarithms, **A**lgebraic expressions like powers of $x$, **T**rig functions, **E**xponential functions). If the integrand contains an inverse trig function, let $u$ be that piece of the integrand. If there is no inverse function in the integrand but there is a logarithm, let $u$ be the logarithm. If there are no inverse functions or logs but there is an algebraic expression (e.g. a power of $x$), let $u$ be the algebraic expression, etc. Don't forget that you may have to integrate by parts more than once.

Example. To find $\int 4x\ln(x)\;dx$ let $u = \ln(x)$ and $dv = 4x\;dx$. Then $u = \ln(x) \Rightarrow du = \frac{1}{x}\;dx$ and $dv = 4x\;dx \Rightarrow v = \int 4x\;dx = 2x^2$, so $\int \underbrace{4x\ln(x)\;dx}_{u\;dv} = \underbrace{2x^2\ln(x)}_{uv} - \int \underbrace{2x\;dx}_{v\;du} = 2x^2\ln(x) - x^2$

V. If the integrand contains a term of the form $\sqrt{a^2-u^2}$, $\sqrt{a^2-u^2}$, or $\sqrt{a^2+u^2}$, a trigonometric substitution may be helpful.

1. For integrands involving $\sqrt{a^2-u^2}$ let $u = a\sin(t)$. E.g., for $\int \frac{1}{x^2\sqrt{25-x^2}}\;dx$, let $x = 5\sin(t)$.
2. For integrands involving $\sqrt{u^2-a^2}$ let $u = a\sec(t)$. E.g., for $\int \frac{1}{\sqrt{x^2-9}}\;dx$, let $x = 3\sec(t)$.
3. For integrands involving $\sqrt{a^2+u^2}$ let $u = a\tan(t)$. E.g., for $\int \frac{\sqrt{4x^2+9}}{x^4}\;dx$, let $2x = 3\tan(t)$.

VI. If the integrand is a proper rational function, try a partial fraction decomposition.

Example. To find $\int \frac{x^2-2x+1}{x^4+x^2}\;dx$, use partial fractions:

$$\begin{aligned}
\frac{x^2-2x+1}{x^4+x^2} &= \frac{x^2-2x+1}{x^2\left(x^2+1\right)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1} \Rightarrow \\
x^2-2x+1 &= \left[\frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}\right]\left(x^4+x^2\right) \\
&= Ax\left(x^2+1\right) + B\left(x^2+1\right) + (Cx+D)\,x^2 \\
&= (A+C)\,x^3 + (B+D)\,x^2 + Ax + B \Rightarrow
\end{aligned}$$

$$\left\{\begin{array}{rcl} A+C &=& 0 \\ B+D &=& 1 \\ A &=& -2 \\ B &=& 1 \end{array}\right.$$

$\Rightarrow A = -2,\; B = 1,\; C = 2,\; D = 0 \Rightarrow$

$$\int \frac{x^2-2x+1}{x^4+x^2}\;dx = \int \left[\frac{-2}{x} + \frac{1}{x^2} + \frac{2x}{x^2+1}\right]\;dx = -2\ln|x| - x^{-1} + \ln\left(x^2+1\right) + c$$