Integration Techniques

I. Basic antidifferentiation rules you should know:

\[
\int x^n \, dx = \frac{1}{n+1} x^{n+1} \quad \text{if } n \neq -1 \\
\int \frac{1}{x} \, dx = \ln |x|
\]

\[
\int e^x \, dx = e^x \\
\int \sin(x) \, dx = -\cos(x)
\]

\[
\int \cos(x) \, dx = \sin(x) \\
\int \tan(x) \, dx = \ln |\sec(x)| = -\ln |\cos(x)|
\]

\[
\int \cot(x) \, dx = \ln |\sin(x)| = -\ln |\csc(x)| \\
\int \sec^2(x) \, dx = \tan(x)
\]

\[
\int \csc^2(x) \, dx = -\cot(x) \\
\int \sec(x) \tan(x) \, dx = \sec(x)
\]

\[
\int \csc(x) \cot(x) \, dx = -\csc(x) \\
\int \frac{1}{x^2 + 1} \, dx = \arctan(x)
\]

\[
\int \frac{1}{\sqrt{1-x^2}} \, dx = \arcsin(x)
\]

II. If your integrand does not appear on the above list, it may be possible to simplify the integrand by expanding it.

Example 1. \( \int (x^2 + 1)^2 \, dx = \int [x^4 + 2x^2 + 1] \, dx \)

Example 2. \( \int \frac{x+1}{x^2} \, dx = \int \left[ \frac{x}{x^2} + \frac{1}{x^2} \right] \, dx = \int \left[ \frac{1}{x} + x^{-2} \right] \, dx \)

III. If it is not possible to simplify your integrand, try a substitution. Rules of thumb for deciding what to choose for \( u \) when using substitution:

1. If an expression appears raised to a power or under a root, let \( u = \) that expression.
   
   Example. To find \( \int \sqrt{2x+5} \, dx \) let \( u = 2x + 5 \).

2. If one part of the integrand is the derivative of a second part of the integrand, let \( u = \) the second part.
   
   Example. To find \( \int 3x^2 \sin (x^3) \, dx \), let \( u = x^3 \).

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IV. If substitution does not work, try integration by parts: \[ \int u \, dv = uv - \int v \, du \]

Rule of thumb for deciding what to choose for \( u \) when using integration by parts: **ILATE** (Inverse trig functions, Logarithms, Algebraic expressions like powers of \( x \), Trig functions, Exponential functions). If the integrand contains an inverse trig function, let \( u \) be that piece of the integrand. If there is no inverse function in the integrand but there is a logarithm, let \( u \) be the logarithm. If there are no inverse functions or logs but there is an algebraic expression (e.g. a power of \( x \)), let \( u \) be the algebraic expression, etc. Don’t forget that you may have to integrate by parts more than once.

Example. To find \( \int \tan^{-1}(x) \, dx \), let \( u = \tan^{-1}(x) \), \( dv = dx \), then \( du = \frac{1}{1+x^2} \, dx \), \( v = x \), so \( \int \tan^{-1}(x) \, dx = x \tan^{-1}(x) - \int \frac{x}{1+x^2} \, dx \).

V. If the integrand contains a term of the form \( \sqrt{a^2 - u^2} \), \( \sqrt{a^2 + u^2} \), or \( \sqrt{a^2 - u^2} \), a trigonometric substitution may be helpful.

1. For integrands involving \( \sqrt{a^2 - u^2} \) let \( u = \sin(t) \). E.g., for \( \int \frac{1}{x^2\sqrt{25-x^2}} \, dx \), let \( x = 5 \sin(t) \).

2. For integrands involving \( \sqrt{a^2 - u^2} \) let \( u = \sec(t) \). E.g., for \( \int \frac{1}{\sqrt{x^2 - 9}} \, dx \), let \( x = 3 \sec(t) \).

3. For integrands involving \( \sqrt{a^2 + u^2} \) let \( u = \tan(t) \). E.g., for \( \int \frac{1}{x^4 \sqrt{4x^2 + 9}} \, dx \), let \( 2x = 3 \tan(t) \).

VI. If the integrand is a proper rational function, try a partial fraction decomposition.

Example. To find \( \int \frac{x^2 - 2x + 1}{x^4 + x^2} \, dx \), use partial fractions:

\[
x^2 - 2x + 1 \quad \frac{x^2 - 2x + 1}{x^4 + x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2 + 1} \quad \Rightarrow \quad x^2 - 2x + 1 = \left[ \frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2 + 1} \right] (x^4 + x^2)
\]

\[
= Ax(x^2 + 1) + B(x^2 + 1) + (Cx + D)x^2
\]

\[
= (A + C)x^3 + (B + D)x^2 + Ax + B \quad \Rightarrow \quad \begin{cases} A + C = 0 \\ B + D = 1 \\ A = -2 \\ B = 1 \end{cases}
\]

\( \Rightarrow A = -2, \ B = 1, \ C = 2, \ D = 0 \Rightarrow \)

\[
\int \frac{x^2 - 2x + 1}{x^4 + x^2} \, dx = \int \left[ \frac{-2}{x} + \frac{1}{x^2} + \frac{2x}{x^2 + 1} \right] \, dx = -2 \ln |x| - x^{-1} + \ln (x^2 + 1) + c
\]