92.231 Calculus III   Practice Exam # 1 Solutions   Spring 2004

Problem #1 (20 points)
Let $\mathbf{a} = \langle 2, 1, -2 \rangle$ and let $\mathbf{b} = \langle 3, 0, 4 \rangle$.

a. Compute $\mathbf{a} - \mathbf{b}$. $\mathbf{a} - \mathbf{b} = \langle 2-3, 1-0, -2-4 \rangle = \boxed{\langle -1, 1, -6 \rangle}$.

b. Compute $\mathbf{a} \cdot \mathbf{b}$. $\mathbf{a} \cdot \mathbf{b} = 2 \cdot 3 + 1 \cdot 0 + (-2) \cdot 4 = \boxed{-2}$.

c. Compute $\mathbf{a} \times \mathbf{b}$. $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -2 \\ 3 & 0 & 4 \end{vmatrix} = \mathbf{i}\,[(1)(4) - (0)(-2)] + \mathbf{j}\,[(-2)(3) - (2)(4)] + \mathbf{k}\,[(2)(0) - (1)(3)] =$
$\boxed{\langle 4, -14, -3 \rangle}$.

d. Compute $\text{proj}_{\mathbf{a}}\mathbf{b}$ (the vector projection of $\mathbf{b}$ onto $\mathbf{a}$). $\text{proj}_{\mathbf{a}}\mathbf{b} = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2}\mathbf{a} = \dfrac{-2}{2^2 + 1^2 + (-2)^2}\mathbf{a} = \boxed{\left\langle -\frac{4}{9}, -\frac{2}{9}, \frac{4}{9} \right\rangle}$.

Problem #2 (10 points)
Find the equation of the plane containing the point $(0, 3, 2)$ and perpendicular to the vector $\langle 2, 1, -2 \rangle$.

The equation of the plane containing the point (x_0, y_0, z_0) and perpendicular to the vector $\langle a, b, c \rangle$ is $a\,(x - x_0) + b\,(y - y_0) + c\,(z - z_0) = 0$. Therefore, the equation of the plane described in this problem is $2\,(x - 0) + 1\,(y - 3) - 2\,(z - 2) = 0$, or $\boxed{2x + y - 2z = -1}$.

Problem #3 (15 points)
Let P denote the point $(1, -1, 2)$ and let S denote the sphere with center at P and radius 3.

a. Let Q denote the point $(3, 0, 0)$. Compute the distance from P to Q. Using the distance formula, we find that $|PQ| = \sqrt{(3-1)^2 + [0-(-1)]^2 + (0-2)^2} = \boxed{3}$.

b. Does Q lie on the sphere S? Why or why not? The sphere S consists of all points whose distance from P is 3. Since $|PQ| = 3$, $\boxed{Q \text{ does lie on } S.}$

c. Find the equation of the sphere S. The equation of a sphere with radius r and center (x_0, y_0, z_0) is $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$, so the equation of S is
$(x-1)^2 + [y-(-1)]^2 + (z-2)^2 = 3^2$, or $\boxed{(x-1)^2 + (y+1)^2 + (z-2)^2 = 9}$.

Problem #4 (15 points)
Let C denote the curve described by the vector function $\mathbf{r}(t) = \left\langle e^{2t}, 2e^{2t}, 2e^{2t} \right\rangle$, $0 \le t \le 1$.

a. Show that $|\mathbf{r}'(t)| = 6e^{2t}$. $\mathbf{r}(t) = \langle e^{2t}, 2e^{2t}, 2e^{2t} \rangle \Rightarrow \mathbf{r}'(t) = \langle 2e^{2t}, 4e^{2t}, 4e^{2t} \rangle$ (because $\dfrac{d}{dt}[e^u] = e^u \dfrac{du}{dt}$). Therefore, $|\mathbf{r}'(t)| = \sqrt{[2e^{2t}]^2 + [4e^{2t}]^2 + [4e^{2t}]^2} = \sqrt{4e^{4t} + 16e^{4t} + 16e^{4t}} = \sqrt{36e^{4t}} = 6e^{2t}$.

b. Find the length of C. The length of C is given by $L = \displaystyle\int_a^b |\mathbf{r}'(t)|\; dt = \underbrace{\int_0^1 6e^{2t}\; dt}_{u=2t,\ du=2dt} = 3e^{2t}\Big|_0^1 =$

$\boxed{3e^2 - 3} \approx 19.167$.

Problem #5 (20 points)
Let C denote the curve described by the vector function $\mathbf{r}(t) = \langle t^2, t, 2-t\rangle$

a. Find $\mathbf{r}'(t)$. $\mathbf{r}(t) = \langle t^2, t, 2-t\rangle \Rightarrow \boxed{\mathbf{r}'(t) = \langle 2t, 1, -1\rangle}$. (Take the derivative of each component.)

b. Show that the point $(4, 2, 0)$ lies on C. We must find a value of t for which $t^2 = 4$, $t = 2$, and $2 - t = 0$. $\boxed{t = 2 \text{ clearly satisfies all three conditions, so the point } (4,2,0) \text{ lies on } C.}$

c. Find parametric equations for the line tangent to C at the point $(4, 2, 0)$. The parametric equations of the line containing the point (x_0, y_0, z_0) and parallel to the vector $\langle a, b, c\rangle$ are $x = x_0 + at, y = y_0 + bt, z = z_0 + ct$. The vector $\mathbf{r}'(t)$ is tangent to C, so $\mathbf{r}'(2) = \langle 2t, 1, -1\rangle|_{t=2} = \langle 4, 1, -1\rangle$ is tangent to C at $(4, 2, 0)$. (From part b we know that the point $(4, 2, 0)$ corresponds to $t = 2$.) The tangent line therefore contains point $(4, 2, 0)$ and is parallel to vector $\langle 4, 1, -1\rangle$, so its parametric equations are $x = 4+4t, y = 2+1t, z = 0+(-1)t$, or $\boxed{x = 4 + 4t, y = 2 + t, z = -t}$.

Problem #6 (20 points)
Let C denote the curve described by the vector function $\mathbf{r}(t) = \langle 1 + \sin(4t), 3t, 2 - \cos(4t)\rangle$.

a. Find the unit tangent vector $\mathbf{T}(t)$. $\mathbf{r}(t) = \langle 1 + \sin(4t), 3t, 2 - \cos(4t)\rangle \Rightarrow \mathbf{r}'(t) = \langle 4\cos(4t), 3, 4\sin(4t)\rangle \Rightarrow$ $|\mathbf{r}'(t)| = \sqrt{[4\cos(4t)]^2 + 3^2 + [4\sin(4t)]^2} = \sqrt{16\cos^2(4t) + 9 + 16\sin^2(4t)} = \sqrt{16\underbrace{\left[\cos^2(4t) + \sin^2(4t)\right]}_{=1} + 9} = \sqrt{16 + 9} = 5$. Therefore, $\mathbf{T}(t) = \frac{1}{|\mathbf{r}'(t)|}\mathbf{r}'(t) = \frac{1}{5}\langle 4\cos(4t), 3, 4\sin(4t)\rangle = \boxed{\left\langle \frac{4}{5}\cos(4t), \frac{3}{5}, \frac{4}{5}\sin(4t)\right\rangle}$.

b. Find the unit normal vector $\mathbf{N}(t)$. $\mathbf{T}(t) = \left\langle \frac{4}{5}\cos(4t), \frac{3}{5}, \frac{4}{5}\sin(4t)\right\rangle \Rightarrow \mathbf{T}'(t) = \left\langle -\frac{16}{5}\sin(4t), 0, \frac{16}{5}\cos(4t)\right\rangle$
$\Rightarrow |\mathbf{T}'(t)| = \sqrt{\left[-\frac{16}{5}\sin(4t)\right]^2 + 0^2 + \left[\frac{16}{5}\cos(4t)\right]^2} = \sqrt{\left(\frac{16}{5}\right)^2 [\cos^2(4t) + \sin^2(4t)]} = \frac{16}{5}$.
$\mathbf{N}(t) = \frac{1}{|\mathbf{T}'(t)|}\mathbf{T}'(t) = \frac{1}{16/5}\left\langle -\frac{16}{5}\sin(4t), 0, \frac{16}{5}\cos(4t)\right\rangle = \boxed{\langle -\sin(4t), 0, \cos(4t)\rangle}$.

c. Find the curvature κ. $\kappa = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{16/5}{5} = \boxed{16/25}$.