92.231 Calculus III Practice Exam # 2 Solutions Spring 2004

Problem #1 (10 points)

Let $z = \sin(xy) + x^2y^3$. Find $\dfrac{\partial z}{\partial x}$, $\dfrac{\partial^2 z}{\partial x^2}$ and $\dfrac{\partial^2 z}{\partial y \partial x}$.

$z = \sin(xy) + x^2y^3 \Rightarrow \dfrac{\partial z}{\partial x} = \cos(xy) \cdot y + (2x) \cdot y^3 = \boxed{y\cos(xy) + 2xy^3}$. (Treat y as a constant.)

$\dfrac{\partial^2 z}{\partial x^2} = \dfrac{\partial}{\partial x}\left[y\cos(xy) + 2xy^3\right] = y\left[-\sin(xy) \cdot y\right] + 2y^3 = \boxed{-y^2\sin(xy) + 2y^3}$

and $\dfrac{\partial^2 z}{\partial y \partial x} = \dfrac{\partial}{\partial y}\left[\dfrac{\partial z}{\partial x}\right] = \dfrac{\partial}{\partial y}\left[\underbrace{y\cos(xy)}_{\text{Use Product Rule}} + 2xy^3\right] = \boxed{\cos(xy) - xy\sin(xy) + 6xy^2}$.

Problem #2 (20 points)

Let $f(x, y) = x^2 - y^4$, let P denote the point $(2, 1)$, and let $\mathbf{v} = \langle 4, -3\rangle$. Find the directional derivative of f at P in the direction of vector $\mathbf{v}$.

$f(x, y) = x^2 - y^4 \Rightarrow \nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle = \langle 2x, -4y^3\rangle \Rightarrow \nabla f(2, 1) = \langle 4, -4\rangle$.
$|\mathbf{v}| = \sqrt{(4)^2 + (-3)^2} = \sqrt{25} = 5$, so a unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \dfrac{1}{5}\langle 4, -3\rangle = \langle 4/5, -3/5\rangle$.
Therefore, the directional derivative of f at P in the direction of vector $\mathbf{v}$ is
$D_{\mathbf{u}}f(2, 1) = \nabla f(2, 1) \cdot \mathbf{u} = \langle 4, -4\rangle \cdot \langle 4/5, -3/5\rangle = \boxed{28/5}$.

Problem #3 (15 points)

Let $f(x, y, z) = x^2y + y^3z$ and let P denote the point $(1, -1, 2)$.

a. Find a vector in the direction in which f increases most rapidly at P.

f increases most rapidly in the direction of $\nabla f(P)$. $f(x, y, z) = x^2y + y^3z \Rightarrow \nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z)\rangle = \langle 2xy, x^2 + 3y^2z, y^3\rangle \Rightarrow \nabla f(1, -1, 2) = \boxed{\langle -2, 7, -1\rangle}$.

b. Find the maximum rate of increase of f at P.

The maximum rate of increase of f at P is $|\nabla f(P)| = |\langle -2, 7, -1\rangle| = \sqrt{(-2)^2 + 7^2 + (-1)^2} = \sqrt{54} = \boxed{3\sqrt{6}}$.

Problem #4 (20 points)

The radius r and the height h of a right circular cylinder change with time. At a certain instant the dimensions are $r = 10$ cm and $h = 20$ cm, the radius r is increasing at a rate of 2 cm/sec, and the height h is decreasing at a rate of 3 cm/sec. At what rate is the volume V of the cylinder changing at that instant? (Hint: $V = \pi r^2 h$.)

By the chain rule, $\dfrac{dV}{dt} = \dfrac{\partial V}{\partial r}\dfrac{dr}{dt} + \dfrac{\partial V}{\partial h}\dfrac{dh}{dt} = 2\pi rh\dfrac{dr}{dt} + \pi r^2\dfrac{dh}{dt} =$
$2\pi(10\text{ cm})(20\text{ cm})(2\text{ cm/sec}) + \pi(10\text{ cm})^2(-3\text{ cm/sec}) = \boxed{500\pi\text{ cm}^3/\text{sec}}$.

Problem #5 (20 points)

Let $f(x,y) = x^3 - 3xy + \frac{1}{2}y^2$.

a. Show that $(0,0)$ and $(3,9)$ are the only critical points of f.

The critical points of f are the points (x,y) that satisfy the equations $f_x(x,y) = 0$ and $f_y(x,y) = 0$. $f(x,y) = x^3 - 3xy + \frac{1}{2}y^2 \Rightarrow f_x(x,y) = 3x^2 - 3y,\ f_y(x,y) = -3x + y$.
$f_x(x,y) = 0 \Rightarrow 3x^2 - 3y = 0 \Rightarrow y = x^2$.
$f_y(x,y) = 0 \Rightarrow -3x + y = 0 \Rightarrow -3x + x^2 = 0$ (since $y = x^2$) $\Rightarrow x(-3+x) = 0 \Rightarrow x = 0$ or $x = 3$.
$x = 0 \Rightarrow y = 0^2 = 0$ and $x = 3 \Rightarrow y = 3^2 = 9$. Therefore, the only critical points are $(0,0)$ and $(3,9)$.

b. Determine whether f has a local max, a local min, or a saddle point at $(0,0)$.

We need to use the Second Partial Derivative test. $f_x(x,y) = 3x^2 - 3y \Rightarrow$
$f_{xx}(x,y) = 6x$ and $f_{xy}(x,y) = -3$. $f_y(x,y) = -3x + y \Rightarrow f_{yy}(x,y) = 1$.
Therefore, $D(x,y) = f_{xx}(x,y)f_{yy}(x,y) - [f_{xy}(x,y)]^2 = (6x)(1) - [-3]^2 = 6x - 9$.
$D(0,0) = 6(0) - 9 = -9$. Since $D < 0$, $\boxed{f \text{ has a saddle point at } (0,0)}$.

c. Determine whether f has a local max, a local min, or a saddle point at $(3,9)$.

As shown above, $D(x,y) = 6x - 9$. Therefore, $D(3,9) = 6(3) - 9 = 9$. Since $D > 0$ and $f_{xx}(3,9) = 6(3) > 0$, $\boxed{f \text{ has a local minimum at } (3,9)}$.

Problem #6 (15 points)

Find the volume of the solid lying under the circular paraboloid $z = x^2 + y^2$ and above the rectangle $R = [-2,2] \times [-3,3]$.

$$\text{Volume} = \iint_R (x^2+y^2)\ dA = \int_{-2}^{2}\int_{-3}^{3} \left(x^2+y^2\right)\ dy\ dx = \int_{-2}^{2} \left[x^2y + \frac{y^3}{3}\right]\Bigg|_{-3}^{3}\ dx$$

$$= \int_{-2}^{2} \left[\left(3x^2 + \frac{3^3}{3}\right) - \left(-3x^2 + \frac{(-3)^3}{3}\right)\right] = \int_{-2}^{2} \left(6x^2 + 18\right)\ dx = 2x^3 + 18x\Big|_{-2}^{2} =$$

$$\left(2(2^3) + 18(2)\right) - \left(2(-2)^3) + 18(-2)\right) = \boxed{104}.$$