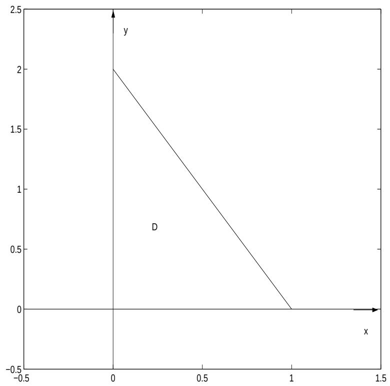
92.231 Calculus III   Exam # 3 Solutions   Spring 2003

Problem #1 (20 points)

Let D be the triangular region with vertices $(0,0), (0,2),$ and $(1,0)$.

a. Sketch the region D.



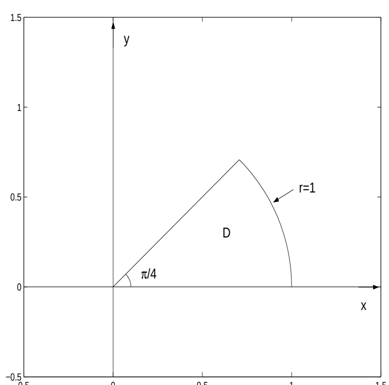
b. Evaluate $\displaystyle\iint_D 3x\, dA$.

The equation of the line through the points $(0,2)$ and $(1,0)$ is $y = 2-2x$. Therefore, $\displaystyle\iint_D 3x\, dA =$

$$\int_0^1 \int_0^{2-2x} 3x\, dy\, dx = \int_0^1 3xy|_{y=0}^{y=2-2x}\, dx = \int_0^1 3x\,(2-2x)\, dx = \int_0^1 6x - 6x^2\, dx = 3x^2 - 2x^3|_0^1 =$$ $\boxed{1}$.

Problem #2 (20 points)

Let D be the region in the first quadrant bounded by the x axis, the line $y = x$, and the circle $x^2 + y^2 = 1$. (See the figure below). Use polar coordinates to evaluate $\displaystyle\iint_D 3y\, dA$.



In polar coordinates, $y = r\sin(\theta)$ and $dA = r\, dr\, d\theta$. Therefore, $\displaystyle\iint_D 3y\, dA = \int_0^{\pi/4}\int_0^1 3r\sin(\theta)\ (r\ dr\ d\theta) =$

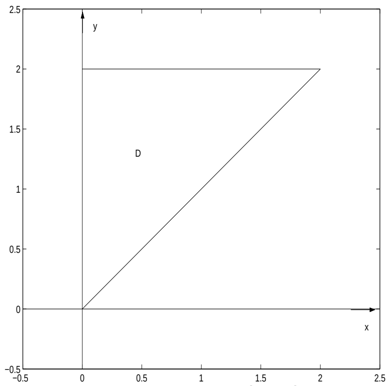
$$\int_0^{\pi/4}\int_0^1 3r^2\sin(\theta)\ dr\ d\theta = \int_0^{\pi/4} r^3\sin(\theta)|_{r=0}^{r=1}\ d\theta = \int_0^{\pi/4}\sin(\theta)\ d\theta = -\cos(\theta)|_0^{\pi/4} =$$

$$-\cos(\pi/4) - (-\cos(0)) = \boxed{1 - \frac{\sqrt{2}}{2}} \approx 0.29289.$$

Problem #3 (20 points)

Find the volume of the solid above the xy plane, below the surface $z = 9 - x^2$, and bounded on the sides by the planes $x = 0$, $y = x$, and $y = 2$.

Volume $V = \iint_D 9 - x^2 \; dA$, where D is the region in the xy plane bounded by $x = 0$, $y = x$, and $y = 2$:



Therefore, $V = \int_0^2 \int_x^2 9 - x^2 \; dy \; dx = \int_0^2 (9 - x^2)\, y\big|_{y=x}^{y=2} \; dx = \int_0^2 (9 - x^2)\, 2 - (9 - x^2)\, x \; dx = \int_0^2 18 - 2x^2 - 9x + x^3 \; dx = 18x - \frac{2x^3}{3} - \frac{9x^2}{2} + \frac{x^4}{4}\Big|_0^2 = \boxed{50/3}$.

Problem #4 (20 points)

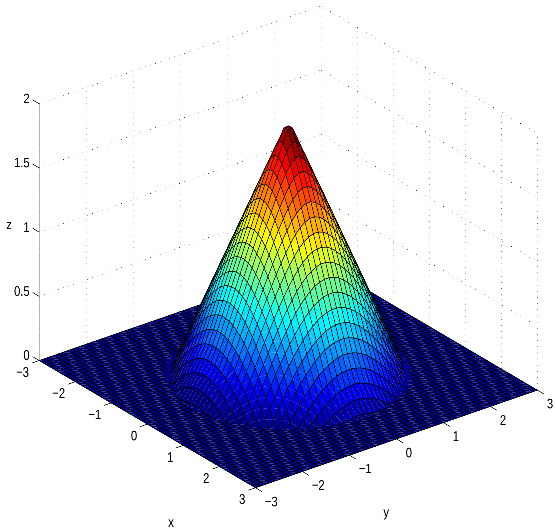
Find the work done by the force field $\mathbf{F}(x, y, z) = \langle -z, y, x \rangle$ in moving a particle from the point $(3, 0, 0)$ to the point $(0, \pi/2, 3)$ along the helix $x = 3\cos(t),\; y = t,\; z = 3\sin(t)$.

The helix is given by the vector function $\mathbf{r}(t) = \langle 3\cos(t), t, 3\sin(t) \rangle$. The initial point on the helix, $(3, 0, 0)$, corresponds to the value $t = 0$, and the terminal point $(0, \pi/2, 3)$ corresponds to the value $t = \pi/2$. The work is given by $W = \int_0^{\pi/2} \mathbf{F}\left(\mathbf{r}(t)\right) \cdot \mathbf{r}'(t) \; dt$

$= \int_0^{\pi/2} \langle -3\sin(t), t, 3\cos(t) \rangle \cdot \langle -3\sin(t), 1, 3\cos(t) \rangle \; dt = \int_0^{\pi/2} \left[9\sin^2(t) + t + 9\cos^2(t) \right] \; dt = \int_0^{\pi/2} [9 + t] \; dt = 9t + \frac{t^2}{2}\Big|_0^{\pi/2} = \boxed{\frac{9\pi}{2} + \frac{\pi^2}{8}}$.

Problem #5 (20 points)

Evaluate $\displaystyle\iiint_E 2z\ dV$, where E is the region above the plane $z = 0$ and below the cone $z = 2 - \sqrt{x^2 + y^2}$.
(Hint: Show that the curve of intersection of the plane $z = 0$ and the cone $z = 2 - \sqrt{x^2 + y^2}$ is the circle $x^2 + y^2 = 4$ in the xy plane.)



Because of the symmetry of E about the z axis, we use cylindrical coordinates. The curve of intersection of the plane $z = 0$ and the cone $z = 2 - \sqrt{x^2 + y^2}$ is the circle $x^2 + y^2 = 4$ in the xy plane: $z = 0$ and $z = 2 - \sqrt{x^2 + y^2} \Rightarrow 0 = 2 - \sqrt{x^2 + y^2} \Rightarrow \sqrt{x^2 + y^2} = 2 \Rightarrow x^2 + y^2 = 4$. Therefore, the projection of E onto the xy plane is the set of points on and inside the circle $x^2 + y^2 = 4$, or $r = 2$ in cylindrical coordinates. In cylindrical coordinates, $dV = r\ dz\ dr\ d\theta$ and $\sqrt{x^2 + y^2} = r$. It follows that $\displaystyle\iiint_E 2z\ dV = \int_0^{2\pi}\int_0^2\int_0^{2-r} 2z\ (r\ dz\ dr\ d\theta) = \int_0^{2\pi}\int_0^2 rz^2\Big|_{z=0}^{z=2-r}\ dr\ d\theta =$
$\displaystyle\int_0^{2\pi}\int_0^2 r\,(2-r)^2\ dr\ d\theta = \int_0^{2\pi}\int_0^2 r^3 - 4r^2 + 4r\ dr\ d\theta = \int_0^{2\pi} \frac{r^4}{4} - \frac{4r^3}{3} + 2r^2\Bigg|_{r=0}^{r=2} d\theta = \int_0^{2\pi} 4/3\ d\theta =$
$4\theta/3\big|_0^{2\pi} = \boxed{8\pi/3}$.