1. Let C be the curve described by the vector function $\mathbf{r}(t) = \langle \sin(t), 2t, \cos(t) \rangle$.

   a. Find $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$.

   $\mathbf{r}(t) = \langle \sin(t), 2t, \cos(t) \rangle \Rightarrow \boxed{\mathbf{r}'(t) = \langle \cos(t), 2, -\sin(t) \rangle, \mathbf{r}''(t) = \langle -\sin(t), 0, -\cos(t) \rangle}$

   b. Find a vector tangent to C at the point $(0, 0, 1)$.

   The point $(0, 0, 1)$ corresponds to the value $t = 0$: $\mathbf{r}(t) = \langle \sin(t), 2t, \cos(t) \rangle = \langle 0, 0, 1 \rangle \Rightarrow \sin(t) = 0,\ 2t = 0,\ \cos(t) = 1 \Rightarrow t = 0$. The vector $\mathbf{r}'(a)$ is tangent to C at the point corresponding to $t = a$, so $\mathbf{r}'(0) = \boxed{\langle 1, 2, 0 \rangle}$ is tangent to C at $(0, 0, 1)$.

   c. Find parametric equations of the line tangent to C at the point $(0, 0, 1)$.

   The line tangent to C at the point $(0, 0, 1)$ is parallel to the vector $\langle 1, 2, 0 \rangle$ and passes through the point $(0, 0, 1)$. Therefore, the line is described by the parametric equations $x = 0 + 1t,\ y = 0 + 2t,\ z = 1 + 0t$, or $\boxed{x = t,\ y = 2t,\ z = 1}$.

2. Let L_1 be the line given by the parametric equations $x = 1 + t,\ y = 1 + t,\ z = 1 + 3t$ and let L_2 be the line given by the parametric equations $x = 2 + t,\ y = -t,\ z = -t$.

   a. Show that the point $(1, 1, 1)$ lies on both L_1 and L_2.

   To show that $(1, 1, 1)$ lies on L_1 we must show that there is a value of t for which $1 = 1 + t,\ 1 = 1 + t$, and $1 = 1 + 3t$. Clearly, $t = 0$ satisfies all three equations, so $(1, 1, 1)$ lies on L_1. To show that $(1, 1, 1)$ lies on L_2 we must show that there is a value of t for which $1 = 2 + t,\ 1 = -t$, and $1 = -t$. Clearly, $t = -1$ satisfies all three equations, so $(1, 1, 1)$ lies on L_2.

   b. Find the equation of the plane containing L_1 and L_2.

   To find the equation of a plane we need a point on the plane and a vector $\mathbf{v}$ perpendicular to the plane. As shown in part a, the point $(1, 1, 1)$ lies in the plane containing L_1 and L_2. A vector perpendicular to the plane containing L_1 and L_2 must be perpendicular to both L_1 and L_2, so we can take $\mathbf{v}$ to be the cross product of the direction vectors for L_1 and L_2. A direction vector for L_1 is $\mathbf{a} = \langle 1, 1, 3 \rangle$, and a direction vector for L_2 is $\mathbf{b} = \langle 1, -1, -1 \rangle$. (The coefficients of t in the parametric equations are the components of the direction vector.) $\mathbf{v} = \mathbf{a} \times \mathbf{b} = \langle 2, 4, -2 \rangle$ is perpendicular to the plane containing L_1 and L_2, and the plane contains the point $(1, 1, 1)$. Therefore, the equation of the plane is $2(x - 1) + 4(y - 1) - 2(z - 1) = 0$, or $\boxed{x + 2y - z = 2}$.

3. Suppose $z = f(x, y)$, $x = g(t)$, and $y = h(t)$. Find $\left.\dfrac{dz}{dt}\right|_{t=1}$ if $g(1) = 3,\ h(1) = 2,\ g'(1) = -1,\ h'(1) = 2,\ f_x(3, 2) = 4$, and $f_y(3, 2) = -2$.

   The Chain Rule says that $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt}$. Therefore, $\left.\dfrac{dz}{dt}\right|_{t=1} = (4)(-1) + (-2)(2) = \boxed{-8}$.

4. Let $f(x, y, z) = x + y\cos(z)$, let P denote the point $(1, 2, 0)$, and let $\mathbf{a} = \langle 2, 1, -2\rangle$.

   a. Find the directional derivative of f at P in the direction $\mathbf{a}$.

   $f(x, y, z) = x + y\cos(z) \Rightarrow \nabla f(x, y, z) = \langle f_x, f_y, f_z\rangle = \langle 1, \cos(z), -y\sin(z)\rangle$. Therefore, $\nabla f(1, 2, 0) = \langle 1, 1, 0\rangle$. $\mathbf{a} = \langle 2, 1, -2\rangle \Rightarrow |\mathbf{a}| = \sqrt{2^2 + 1^2 + (-2)^2} = 3$. Therefore, a unit vector in the direction of $\mathbf{a}$ is $\mathbf{u} = \frac{1}{|\mathbf{a}|}\mathbf{a} = \langle 2/3, 1/3, -2/3\rangle$ and the directional derivative of f at P in the direction $\mathbf{a}$ is $D_{\mathbf{u}} f(1, 2, 0) = \nabla f(1, 2, 0) \cdot \mathbf{u} = \langle 1, 1, 0\rangle \cdot \langle 2/3, 1/3, -2/3\rangle = \boxed{1}$.

   b. Find a vector in the direction in which f increases most rapidly at P.

   f increases most rapidly in the direction $\nabla f(P) = \boxed{\langle 1, 1, 0\rangle}$.

5. Evaluate $\displaystyle\iint_D (x^2 + y^2)\ dA$, where D is the triangular region with vertices $(-1, 0)$, $(0, 1)$ and $(1, 0)$.

The left boundary of D is the line $y = x + 1$, or $x = y - 1$; the right boundary of D is the line $y = 1 - x$, or $x = 1 - y$. Express the given double integral as an iterated integral, integrating first with respect to x: $\displaystyle\iint_D (x^2 + y^2)\ dA = \int_0^1 \int_{y-1}^{1-y} (x^2 + y^2)\ dx\ dy =$

$$\int_0^1 \frac{x^3}{3} + y^2 x\Bigg|_{x=y-1}^{x=1-y} dy = \int_0^1 \left\{\left[\frac{(1-y)^3}{3} + y^2(1-y)\right] - \left[\frac{(y-1)^3}{3} + y^2(y-1)\right]\right\} dy =$$

$$\int_0^1 \left[\frac{2(1-y)^3}{3} + 2y^2 - 2y^3\right] dy = -\frac{(1-y)^4}{6} + \frac{2y^3}{3} - \frac{y^4}{2}\Bigg|_0^1 = \boxed{1/3}.$$

6. Let $\mathbf{F}(x, y, z) = \langle 2xy, x^2 + 2yz, y^2\rangle$ and let $f(x, y, z) = x^2 y + y^2 z$.

   a. Show that f is a potential for $\mathbf{F}$.

   To show that f is a potential for $\mathbf{F}$, we must show that $\nabla f(x, y, z) = \mathbf{F}(x, y, z)$. $\nabla f(x, y, z) = \langle f_x, f_y, f_z\rangle = \langle 2xy, x^2 + 2yz, y^2\rangle = \mathbf{F}$, so f is a potential for $\mathbf{F}$.

   b. Evaluate $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the helix $x = \sin(t),\ y = \cos(t),\ z = t,\ 0 \le t \le \pi$.

   The starting point of C is $(\sin(0), \cos(0), 0) = (0, 1, 0)$ and the terminal point of C is $(\sin(\pi), \cos(\pi), \pi) = (0, -1, \pi)$. By the Fundamental Theorem for Line Integrals, $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r} = f(0, -1, \pi) - f(0, 1, 0) = \left[0^2(-1) + (-1)^2\pi\right] - \left[0^2(1) + (1)^2(0)\right] = \boxed{\pi}$.

7. Evaluate the integral $\displaystyle\iiint\limits_E z\ dV$,

where E is the region between the spheres $x^2+y^2+z^2=1$ and $x^2+y^2+z^2=9$.

Use spherical coordinates. In spherical coordinates, $z=\rho\cos(\phi)$ and $dV=\rho^2\sin(\phi)\ d\rho\ d\phi\ d\theta$. The equations of the spheres forming the boundaries of E are $\rho=1$ and $\rho=3$ in spherical coordinates. Therefore, $\displaystyle\iiint\limits_E z\ dV = \int_0^{2\pi}\int_0^{\pi}\int_1^3 \rho\cos(\phi)\left(\rho^2\sin(\phi)\ d\rho\ d\phi\ d\theta\right) =$

$$\int_0^{2\pi}\int_0^{\pi}\int_1^3 \rho^3\cos(\phi)\sin(\phi)\ d\rho\ d\phi\ d\theta = \int_0^{2\pi}\int_0^{\pi} \frac{\rho^4}{4}\cos(\phi)\sin(\phi)\Big|_{\rho=1}^{\rho=3}\ d\phi\ d\theta =$$

$$\int_0^{2\pi}\underbrace{\int_0^{\pi} 20\cos(\phi)\sin(\phi)\ d\phi}_{u=\sin(\phi)}\ d\theta = \int_0^{2\pi} 10\sin^2(\phi)\Big|_{\phi=0}^{\phi=\pi}\ d\theta = \int_0^{2\pi} 0\ d\theta = \boxed{0}.$$

8. Let $\mathbf{F}(x,y,z)=\langle yz+y, xz+x, xy+1\rangle$.

a. Find $\nabla\cdot\mathbf{F}$ $(=\operatorname{div}(\mathbf{F}))$

$\mathbf{F}(x,y,z)=\langle yz+y, xz+x, xy+1\rangle \Rightarrow \nabla\cdot\mathbf{F}(x,y,z) = \dfrac{\partial}{\partial x}(yz+y)+\dfrac{\partial}{\partial y}(xz+x)+\dfrac{\partial}{\partial z}(xy+1) = \boxed{0}$.

b. Find $\nabla\times\mathbf{F}$ $(=\operatorname{curl}(\mathbf{F}))$

$$\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ yz+y & xz+x & xy+1 \end{vmatrix} = (x-x)\ \mathbf{i}+(y-y)\ \mathbf{j}+(z-z)\ \mathbf{k}=\mathbf{0}.$$

c. Show that $\mathbf{F}$ is conservative.

Since $\nabla\times\mathbf{F}=\mathbf{0}$, $\mathbf{F}$ is conservative.

d. Find a potential for $\mathbf{F}$.

We must find a function f such that $\nabla f=\mathbf{F}$. $\nabla f=\mathbf{F}\Rightarrow\langle f_x,f_y,f_z\rangle=\langle yz+y,xz+x,xy+1\rangle\Rightarrow$ $f_x=yz+y$, $f_y=xz+x$, and $f_z=xy+1$. $f_x=yz+y\Rightarrow f=\int yz+y\ dx=$ $xyz+xy+g(y,z)\Rightarrow f_y=xz+x+g_y$. However, $f_y=xz+x$, so $xz+x+g_y=xz+x\Rightarrow$ $g_y=0\Rightarrow g(y,z)=\int 0\ dy=0+h(z)\Rightarrow f=xyz+xy+h(z)$. Therefore, $f_z=xy+h'(z)$. But $f_z=xy+1$, so $xy+h'(z)=xy+1\Rightarrow h'(z)=1\Rightarrow h(z)=\int 1\ dz=z+c$. It follows that the potential f is given by $\boxed{f(x,y,z)=xyz+xy+z+c}$.

9. Let S denote the part of the surface $z = 9 - x^2 - y^2$ above the xy plane and let $\mathbf{F} = \langle x, y, z \rangle$.

a. Find a vector perpendicular to S at the point (x, y, z) having positive $\mathbf{k}$ component. (You should get $\langle 2x, 2y, 1 \rangle$ or a positive multiple of this.)

The surface S can be described by the parametric equations $x = x, y = y, z = 9 - x^2 - y^2$. Let $\mathbf{r}(x,y) = \langle x, y, 9 - x^2 - y^2 \rangle$. The vector $\mathbf{r}_x \times \mathbf{r}_y$ is perpendicular to S. $\mathbf{r}(x,y) = \langle x, y, 9 - x^2 - y^2 \rangle \Rightarrow \mathbf{r}_x = \langle 1, 0, -2x \rangle$ and $\mathbf{r}_y = \langle 0, 1, -2y \rangle$. Therefore, $\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{vmatrix} = 2x\ \mathbf{i} + 2y\ \mathbf{j} + 1\ \mathbf{k} = \boxed{\langle 2x, 2y, 1 \rangle}$.

b. Evaluate $\displaystyle\iint_S \mathbf{F} \cdot d\mathbf{S}$.

Using the parametrization for S from part a, we see that $\mathbf{F} = \langle x, y, 9 - x^2 - y^2 \rangle$ on S. S intersects the xy plane ($z = 0$) in the circle $9 - x^2 - y^2 = 0$, or $x^2 + y^2 = 9$. Therefore, $\displaystyle\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F} \cdot (\mathbf{r}_x \times \mathbf{r}_y)\ dA$, where D is the region inside the circle $x^2 + y^2 = 9$.

We use polar coordinates to evaluate this double integral.

$\displaystyle\iint_D \mathbf{F} \cdot (\mathbf{r}_x \times \mathbf{r}_y)\ dA = \iint_D \langle x, y, 9 - x^2 - y^2 \rangle \cdot \langle 2x, 2y, 1 \rangle\ dA = \iint_D 9 + x^2 + y^2\ dA =$

$\displaystyle\int_0^{2\pi} \int_0^3 9 + r^2\ (r\ dr\ d\theta) = \int_0^{2\pi} \int_0^3 9r + r^3\ dr\ d\theta = \int_0^{2\pi} \frac{9r^2}{2} + \frac{r^4}{4} \Big|_{r=0}^{r=3} d\theta = \int_0^{2\pi} \frac{243}{4}\ d\theta =$ $\boxed{243\pi/2}$.

10. Let $\mathbf{F}(x,y,z) = \langle xz, yz, xy \rangle$ and let S denote the hemisphere $x^2 + y^2 + z^2 = 1,\ z \geq 0$. **Use Stokes's Theorem** to evaluate $\displaystyle\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\ dS$, where $\mathbf{n}$ is the unit outer normal vector.

The conclusion of Stokes's Theorem is that $\displaystyle\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\ dS = \int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the curve that forms the boundary of S. In this problem, the intersection of S with the xy plane ($z = 0$) is the circle $x^2 + y^2 = 1$, so this circle is the curve C. We can parametrize C by the equations $x = \cos(t),\ y = \sin(t),\ z = 0,\ 0 \leq t \leq 2\pi$. Let $\mathbf{r}(t) = \langle \cos(t), \sin(t), 0 \rangle$. On C, $\mathbf{F} = \langle [\cos(t)](0), [\sin(t)](0), \cos(t)\sin(t) \rangle = \langle 0, 0, \cos(t)\sin(t) \rangle$. Therefore, $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r} =$

$\displaystyle\int_0^{2\pi} \mathbf{F} \cdot \mathbf{r}'\ dt = \int_0^{2\pi} \langle 0, 0, \cos(t)\sin(t) \rangle \cdot \langle -\sin(t), \cos(t), 0 \rangle\ dt = \int_0^{2\pi} 0\ dt = \boxed{0}$.