92.231 Calculus III   Exam # 2   Spring 2004

Please write all answers and all work on the blue book provided.
PLEASE SHOW ALL WORK! You will not receive full credit if you do not show your work.

Problem #1 (10 points)

a. Find the cylindrical coordinates of the point P whose rectangular coordinates are $(x, y, z) = (-1, 1, 1)$.

$r = \sqrt{x^2 + y^2} = \sqrt{(-1)^1 + 1^2} = \sqrt{2}$. Since $x < 0$, $\theta = \tan^{-1}(y/x) + \pi = \tan^{-1}(1/(-1)) + \pi = -\pi/4 + \pi = 3\pi/4$. Therefore, the cylindrical coordinates of P are $\boxed{(r, \theta, z) = \left(\sqrt{2}, 3\pi/4, 1\right)}$.

b. Find the rectangular coordinates of the point whose spherical coordinates are $(\rho, \theta, \phi) = (2, \pi, \pi/2)$.

$x = \rho \sin(\phi) \cos(\theta) = 2 \sin(\pi/2) \cos(\pi) = 2(1)(-1) = -2$. $y = \rho \sin(\phi) \sin(\theta) = 2 \sin(\pi/2) \sin(\pi) = 2(1)(0) = 0$. $z = \rho \cos(\phi) = 2 \cos(\pi/2) = 2(0) = 0$. Therefore, the rectangular coordinates of Q are $\boxed{(x, y, z) = (-2, 0, 0).}$

Problem #2 (25 points)

Let $f(x, y, z) = x^2 + yz$, let P denote the point $(1, 2, 3)$, and let Q denote the point $(2, 4, 5)$.

a. Find the directional derivative of f at P in the direction of vector $\overrightarrow{PQ}$.

$\overrightarrow{PQ} = \langle 2-1, 4-2, 5-3 \rangle = \langle 1, 2, 2 \rangle$. $\left|\overrightarrow{PQ}\right| = \sqrt{1^2 + 2^2 + 2^2} = 3$. Therefore, a unit vector in the direction of $\overrightarrow{PQ}$ is $\mathbf{u} = \frac{1}{3} \langle 1, 2, 2 \rangle = \left\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right\rangle$.
$f(x, y, z) = x^2 + yz \Rightarrow \nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle = \langle 2x, z, y \rangle \Rightarrow$
$\nabla f(1, 2, 3) = \langle 2, 3, 2 \rangle$. Therefore, $D_{\mathbf{u}} f(P) = \nabla f(1, 2, 3) \cdot \mathbf{u} = \langle 2, 3, 2 \rangle \cdot \left\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right\rangle$
$= 2(1/3) + 3(2/3) + 2(2/3) = \boxed{4.}$

b. Find a vector in the direction in which f increases most rapidly at P.

f increases most rapidly in the direction of $\nabla f(P) = \boxed{\langle 2, 3, 2 \rangle.}$

c. Find the maximum rate of increase of f at P.

The maximum rate of increase of f at P is $|\nabla f(P)| = |\langle 2, 3, 2 \rangle| = \sqrt{2^2 + 3^2 + 2^2} = \boxed{\sqrt{17}.}$

Problem #3 (20 points)

Let $f(x,y,z) = 4x^2 + y^2 - z^2$.

a. Find a vector perpendicular to the level surface $f(x,y,z) = 1$ at the point $(1,-1,2)$.

$\nabla f(1,-1,2)$ is perpendicular to the level surface $f(x,y,z) = 1$ at the point $(1,-1,2)$. $f(x,y,z) = 4x^2+y^2-z^2 \Rightarrow \nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle = \langle 8x, 2y, -2z\rangle \Rightarrow \nabla f(1,-1,2) =$ $\boxed{\langle 8,-2,-4\rangle \text{ is perpendicular to the given surface at the given point.}}$

b. Find the equation of the plane tangent to the surface $f(x,y,z) = 1$ at the point $(1,-1,2)$.

$\langle 8,-2,-4\rangle$ is perpendicular to the surface $f(x,y,z) = 1$, so this vector is also perpendicular to the tangent plane. The equation of the plane with normal vector $\langle 8,-2,-4\rangle$ that contains the point $(1,-1,2)$ is $8(x-1) - 2(y-(-1)) - 4(z-2) = 0$, or $\boxed{4x - y - 2z = 1.}$

Problem #4 (25 points)

Let $f(x,y) = 6xy - x^3 - 8y^3$.

a. Show that $(0,0)$ and $(1,1/2)$ are the only critical points of f.

The critical points of f are the points (x,y) that satisfy the equations $f_x(x,y) = 0$ and $f_y(x,y) = 0$. $f(x,y) = 6xy - x^3 - 8y^3 \Rightarrow f_x(x,y) = 6y - 3x^2,\ f_y(x,y) = 6x - 24y^2$.
$f_y(x,y) = 0 \Rightarrow 6x - 24y^2 = 0 \Rightarrow x = 4y^2$.
$f_x(x,y) = 0 \Rightarrow 6y - 3x^2 = 0 \Rightarrow 6y - 3\left(4y^2\right)^2 = 0$ (since $x = 4y^2$) $\Rightarrow 6y\left(1 - 8y^3\right) = 0 \Rightarrow y = 0$ or $y^3 = 1/8 \Rightarrow y = 1/2$.
$y = 0 \Rightarrow x = 4(0)^2 = 0$ and $y = 1/2 \Rightarrow x = 4(1/2)^2 = 1$. Therefore, the only critical points are $(0,0)$ and $(1,1/2)$.

b. Determine whether f has a local max, a local min, or a saddle point at $(0,0)$.

We need to use the Second Partial Derivative test. $f_x(x,y) = 6y - 3x^2 \Rightarrow$ $f_{xx}(x,y) = -6x$ and $f_{xy}(x,y) = 6$. $f_y(x,y) = 6x - 24y^2 \Rightarrow f_{yy}(x,y) = -48y$.
Therefore, $D(x,y) = f_{xx}(x,y)f_{yy}(x,y) - \left[f_{xy}(x,y)\right]^2 = (-6x)(-48y) - (6)^2 = 288xy - 36$.
$D(0,0) = 288(0)(0) - 36 = -36$. Since $D < 0$, $\boxed{f \text{ has a saddle point at } (0,0)}$.

c. Determine whether f has a local max, a local min, or a saddle point at $(1,1/2)$.

As shown above, $D(x,y) = 288xy - 36$. Therefore, $D(1,1/2) = 288(1)(1/2) - 36 = 108$. Since $D > 0$ and $f_{xx}(1,1/2) = -6(1) < 0$, $\boxed{f \text{ has a local maximum at } (1,1/2)}$.

Problem #5 (20 points)

Evaluate the double integral $\displaystyle\iint_R \left(\frac{2y}{x}\right)\, dA,$

where R is the rectangle $R = \{(x,y) |\ 1 \le x \le 2,\ 0 \le y \le 1\}$.

$$\iint_R \left(\frac{2y}{x}\right) dA = \int_1^2 \int_0^1 \left(\frac{2y}{x}\right) dy\, dx = \int_1^2 \left[\frac{y^2}{x}\Bigg|_{y=0}^{y=1}\right] dx = \int_1^2 \left[\frac{1}{x} - \frac{0}{x}\right] dx = \int_1^2 \frac{1}{x}\, dx = \ln(x)|_1^2 =$$

$\ln(2) - \ln(1) = \boxed{\ln(2).}$