92.231 Calculus III   Exam # 2 Solutions   Spring 2004

Problem #1 (10 points)

a. Find the cylindrical coordinates of the point P whose rectangular coordinates are $(x, y, z) = (-2, 2, 3)$.

$r = \sqrt{x^2+y^2} = \sqrt{(-2)^1+2^2} = \sqrt{8} = 2\sqrt{2}$. Since $x < 0$, $\theta = \tan^{-1}(y/x)+\pi = \tan^{-1}(2/(-2))+\pi = -\pi/4+\pi = 3\pi/4$. Therefore, the cylindrical coordinates of P are $\boxed{(r, \theta, z) = \left(2\sqrt{2}, 3\pi/4, 3\right)}$.

b. Find the rectangular coordinates of the point whose spherical coordinates are $(\rho, \theta, \phi) = (4, \pi/2, \pi/4)$.

$x = \rho\sin(\phi)\cos(\theta) = 4\sin(\pi/4)\cos(\pi/2) = 2(\sqrt{2}/2)(0) = 0$. $y = \rho\sin(\phi)\sin(\theta) = 4\sin(\pi/4)\sin(\pi/2) = 4(\sqrt{2}/2)(1) = 2\sqrt{2}$. $z = \rho\cos(\phi) = 4\cos(\pi/4) = 4(\sqrt{2}/2) = 2\sqrt{2}$. Therefore, the rectangular coordinates of Q are $\boxed{(x, y, z) = (0, 2\sqrt{2}, 2\sqrt{2}).}$

Problem #2 (25 points)

Let $f(x, y, z) = y^2z - 2x$, let P denote the point $(1, 1, 2)$, and let Q denote the point $(3, 4, 8)$.

a. Find the directional derivative of f at P in the direction of vector $\overrightarrow{PQ}$.

$\overrightarrow{PQ} = \langle 3-1, 4-1, 8-2\rangle = \langle 2, 3, 6\rangle$. $\left|\overrightarrow{PQ}\right| = \sqrt{2^2+3^2+6^2} = 7$. Therefore, a unit vector in the direction of $\overrightarrow{PQ}$ is $\mathbf{u} = \frac{1}{7}\langle 2, 3, 6\rangle = \left\langle \frac{2}{7}, \frac{3}{7}, \frac{6}{7}\right\rangle$.
$f(x, y, z) = y^2z - 2x \Rightarrow \nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z)\rangle = \langle -2, 2yz, y^2\rangle \Rightarrow$
$\nabla f(1, 1, 2) = \langle -2, 4, 1\rangle$. Therefore, $D_{\mathbf{u}}f(P) = \nabla f(1, 1, 1)\cdot\mathbf{u} = \langle -2, 4, 1\rangle \cdot \left\langle \frac{2}{7}, \frac{3}{7}, \frac{6}{7}\right\rangle$
$= -2(2/7)+4(3/7)+1(6/7) = \boxed{2.}$

b. Find a vector in the direction in which f increases most rapidly at P.

f increases most rapidly in the direction of $\nabla f(P) = \boxed{\langle -2, 4, 1\rangle.}$

c. Find the maximum rate of increase of f at P.

The maximum rate of increase of f at P is $|\nabla f(P)| = |\langle -2, 4, 1\rangle| = \sqrt{(-2)^2+4^2+1^2} = \boxed{\sqrt{21}.}$

Problem #3 (20 points)

Let $f(x, y, z) = x^2 + y^4 + z^6$.

a. Find a vector perpendicular to the level surface $f(x, y, z) = 3$ at the point $(1, -1, 1)$.

$\nabla f(1, -1, 1)$ is perpendicular to the level surface $f(x, y, z) = 3$ at the point $(1, -1, 1)$. $f(x, y, z) = x^2+y^4+z^6 \Rightarrow \nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z)\rangle = \langle 2x, 4y^3, 6z^5\rangle \Rightarrow \nabla f(1, -1, 1) =$ $\boxed{\langle 2, -4, 6\rangle \text{ is perpendicular to the given surface at the given point.}}$

b. Find the equation of the plane tangent to the surface $f(x, y, z) = 3$ at the point $(1, -1, 1)$.

$\langle 2, -4, 6\rangle$ is perpendicular to the surface $f(x, y, z) = 3$, so this vector is also perpendicular to the tangent plane. The equation of the plane with normal vector $\langle 2, -4, 6\rangle$ that contains the point $(1, -1, 1)$ is $2(x - 1) - 4(y - (-1)) + 6(z - 1) = 0$, or $\boxed{x - 2y + 3z = 6.}$

Problem #4 (25 points)

Let $f(x, y) = 3xy - 2x^3 - 2y^3$.

a. Show that $(0, 0)$ and $(1/2, 1/2)$ are the only critical points of f.

The critical points of f are the points (x, y) that satisfy the equations $f_x(x, y) = 0$ and $f_y(x, y) = 0$. $f(x, y) = 3xy - 2x^3 - 2y^3 \Rightarrow f_x(x, y) = 3y - 6x^2, \ f_y(x, y) = 3x - 6y^2$.
$f_x(x, y) = 0 \Rightarrow 3y - 6x^2 = 0 \Rightarrow y = 2x^2$.
$f_y(x, y) = 0 \Rightarrow 3x - 6y^2 = 0 \Rightarrow 3x - 6\left(2x^2\right)^2 = 0$ (since $y = 2x^2$) $\Rightarrow 3x\left(1 - 8x^3\right) = 0 \Rightarrow x = 0$ or $x^3 = 1/8 \Rightarrow x = 1/2$.
$x = 0 \Rightarrow y = 2(0)^2 = 0$ and $x = 1/2 \Rightarrow y = 2(1/2)^2 = 1/2$. Therefore, the only critical points are $(0, 0)$ and $(1/2, 1/2)$.

b. Determine whether f has a local max, a local min, or a saddle point at $(0, 0)$.

We need to use the Second Partial Derivative test. $f_x(x, y) = 3y - 6x^2 \Rightarrow$
$f_{xx}(x, y) = -12x$ and $f_{xy}(x, y) = 3$. $f_y(x, y) = 3x - 6y^2 \Rightarrow f_{yy}(x, y) = -12y$.
Therefore, $D(x, y) = f_{xx}(x, y)f_{yy}(x, y) - [f_{xy}(x, y)]^2 = (-12x)(-12y) - (3)^2 = 144xy - 9$.
$D(0, 0) = 144(0)(0) - 9 = -9$. Since $D < 0$, $\boxed{f \text{ has a saddle point at } (0, 0)}$.

c. Determine whether f has a local max, a local min, or a saddle point at $(1, 1/2)$.

As shown above, $D(x, y) = 144xy - 9$. Therefore, $D(1/2, 1/2) = 144(1/2)(1/2) - 9 = 27$. Since $D > 0$ and $f_{xx}(1/2, 1/2) = -12(1/2) < 0$, $\boxed{f \text{ has a local maximum at } (1, 1/2)}$.

Problem #5 (20 points)

Evaluate the double integral $\displaystyle\iint_R \left(\frac{4x}{y}\right)\, dA,$

where R is the rectangle $R = \{(x, y)|\ 1 \le x \le 2,\ 1 \le y \le 4\}$.

$$\iint_R \left(\frac{4x}{y}\right) dA = \int_1^2 \int_1^4 \left(\frac{4x}{y}\right) dy\, dx = \int_1^2 \left[4x\ln(y)\Big|_{y=1}^{y=4}\right] dx = \int_1^2 [4x\ln(4) - 4x\ln(1)]\, dx$$
$$= \int_1^2 4x\ln(4)\, dx = 2\ln(4)x^2\Big|_1^2 = 8\ln(4) - 2\ln(4) = \boxed{6\ln(4).}$$