92.236 Differential Equations

The Method of Undetermined Coefficients

The method of undetermined coefficients can be used to find a particular solution y_p of a nonhomogeneous linear d.e. if the d.e. has constant coefficients and the nonhomogeneous term is a polynomial, an exponential, a sine or a cosine, or a sum or product of these.

If the d.e. has variable coefficients and/or the nonhomogeneous term is something other than a polynomial, exponential, sine, cosine, or sum or product of these, you must use another method (e.g. variation of parameters).

1. If the nonhomogeneous term is a polynomial, try guessing a polynomial of the same degree for y_p.
   Example. $y'' - 3y' + 2y = 4x$.
   Nonhomogeneous term: $4x$ (polynomial of degree 1).
   Guess: $y_p = Ax + B$ (a polynomial of degree 1).
   Note: You need the constant term B in your guess for y_p even though there is no constant in the nonhomogeneous term.

2. If the nonhomogeneous term is an exponential function, try guessing an exponential function of the same form for y_p.
   Example. $y'' - 3y' + 2y = 6e^{-x}$.
   Nonhomogeneous term: $6e^{-x}$ (an exponential function).
   Guess: $y_p = Ae^{-x}$ (an exponential function of the same form).

3. If the nonhomogeneous term is a sine or a cosine, try guessing a combination of sine and cosine of the same angular frequency for y_p.
   Example. $y'' - 3y' + 2y = 10\sin(2x)$.
   Nonhomogeneous term: $10\sin(2x)$ (sine function with angular frequency 2).
   Guess: $y_p = A\sin(2x) + B\cos(2x)$ (combination of sine and cosine w. angular frequency 2).
   Note: You need the cosine term $B\cos(2x)$ in your guess for y_p even though there is no cosine in the nonhomogeneous term.

4. If the nonhomogeneous term is a sum of a polynomial, an exponential function, and/or a sine or cosine, try guessing a sum of these functions for y_p.
   Example. $y'' - 3y' + 2y = 4x + 10\sin(2x)$.
   Nonhomogeneous term: sum of a polynomial of degree 1 and a sine with angular frequency 2.
   Guess: $y_p = Ax + B + C\sin(2x) + D\cos(2x)$

5. If the nonhomogeneous term is a product of a polynomial, an exponential function, and/or a sine or cosine, try guessing a product of these functions for y_p.
   Example. $y'' - 3y' + 2y = 36xe^{-x}$.
   Nonhomogeneous term: product of a polynomial of degree 1 and an exponential.
   Guess: $y_p = (Ax + B)e^{-x}$
   Note: You need the Be^{-x} term in your guess for y_p even though there is no such term in the nonhomogeneous term.

IMPORTANT EXCEPTION TO THE ABOVE RULES

If any part of your guess for y_p is a part of the complementary solution y_c, you must multiply that part of your y_p guess by x.

Example 1. $y'' - 3y' + 2y = e^x$.
The homogeneous equation is $y'' - 3y' + 2y = 0$, which has characteristic equation $r^2 - 3r + 2 = 0$. The roots of the characteristic equation are 1 and 2, so the complementary solution is $y_c = c_1 e^x + c_2 e^{2x}$.
Nonhomogeneous term: e^x
Usual guess: $y_p = Ae^x$. A term of this form already appears in y_c, so we must multiply our guess by x.
Correct guess: $y_p = Axe^x$.

Example 2. $y'' - 2y' + y = e^x$.
The homogeneous equation is $y'' - 2y' + y = 0$, which has characteristic equation $r^2 - 2r + 1 = 0$. The root of the characteristic equation is 1 (with multiplicity 2), so the complementary solution is $y_c = c_1 e^x + c_2 xe^x$.
Nonhomogeneous term: e^x.
Usual guess: $y_p = Ae^x$. A term of this form already appears in y_c, so we multiply our guess by x: $y_p = Axe^x$. This new guess is still of the same form as part of y_c, so we must multiply by x again.
Correct guess: $y_p = Ax^2 e^x$.