Engineering Differential Equations
Examples of Forced Motion with Sinusoidal Forcing

1. (Generic Example). Consider a mass-spring system with mass $m = 1$ kg, spring constant $k = 5$ N/m, damping constant $c = 2$ N·sec/m, and external force $F(t) = 17\cos(2t)$ N. Suppose the mass starts from a position 1 m to the right of the equilibrium position with a velocity of 4 m/sec. Find $x(t)$, the position of the mass at time t.

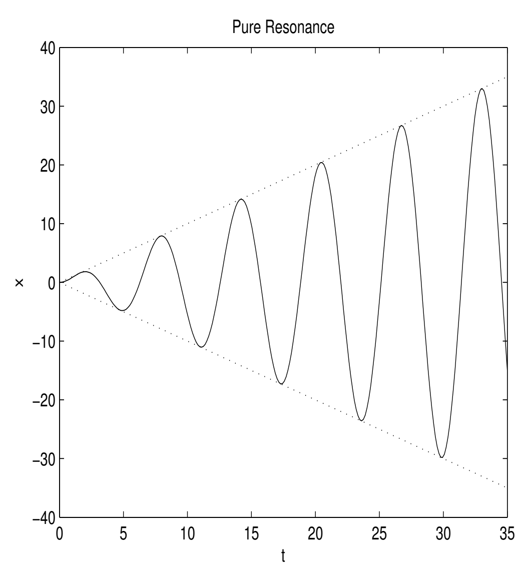
   The model d.e. $mx'' + cx' + kx = F(t)$ becomes $x'' + 2x' + 5x = 17\cos(2t)$. To find x_c, we solve the characteristic equation: $r^2 + 2r + 5 = 0 \Rightarrow r = \dfrac{-2 \pm \sqrt{2^2 - 4(1)(5)}}{2(1)} = -1 \pm 2i$. Therefore, $x_c = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t)$. Since the nonhomogeneous term is $17\cos(2t)$, we guess $x_p = A\cos(2t) + B\sin(2t)$. Substituting into the d.e. and solving for A and B, we find that $x_p = \cos(2t) + 4\sin(2t)$. Thus, $x = x_c + x_p = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t) + \cos(2t) + 4\sin(2t)$. Using the initial conditions $x(0) = 1$ and $x'(0) = 4$, we find that $c_1 = 0$ and $c_2 = -2$, so

   $$x = \underbrace{-2e^{-t}\sin(2t)}_{\text{transient part of solution}} + \underbrace{\cos(2t) + 4\sin(2t)}_{\text{steady-state part of solution}}.$$

   The transient part of the solution approaches 0 as $t \to \infty$. This part of the solution comes from the complementary solution x_c. The steady-state part of the solution is the particular solution x_p. This part of the solution oscillates and does not approaches 0 as $t \to \infty$.

2. (Pure Resonance) Consider a mass-spring system with mass $m = 1$ kg, spring constant $k = 1$ N/m, damping constant $c = 0$ N·sec/m, and external force $F(t) = 2\cos(t)$ N. Suppose the mass starts from rest at the equilibrium position. Find $x(t)$, the position of the mass at time t.

   The model d.e. $mx'' + cx' + kx = F(t)$ becomes $x'' + x = 2\cos(t)$. To find x_c we solve the characteristic equation: $r^2 + 1 = 0 \Rightarrow r = \pm i \Rightarrow x_c = c_1\cos(t) + c_2\sin(t)$. Since the nonhomogeneous term is $2\cos(t)$, we guess $x_p = A\cos(t) + B\sin(t)$. However, both these terms appear in x_c, so we must modify our guess: $x_p = At\cos(t) + Bt\sin(t)$. Substituting into the d.e. and solving for A and B, we find that $x_p = t\sin(t)$. Thus, $x = c_1\cos(t) + c_2\sin(t) + t\sin(t)$. Using the initial conditions $x(0) = 0,\ x'(0) = 0$, we find that $c_1 = 0$ and $c_2 = 0$, so $x = t\sin(t)$. Notice that the amplitude of the oscillations $\to \infty$ as $t \to \infty$. This is an example of **pure resonance**. Pure resonance occurs in an undamped system when the frequency of the forcing term exactly matches the natural frequency of the system $\omega_0 = \sqrt{k/m}$.



3. (Resonance) Consider a mass-spring system with mass $m = 1$ kg, spring constant $k = 1$ N/m, damping constant $c = 0.1$ N·sec/m, and external force $F(t) = \cos(\omega t)$ N, where $\omega = \sqrt{\frac{k}{m} - \frac{c^2}{2m^2}} \approx 0.9975\ \text{sec}^{-1}$. Find the steady-state solution.

   The model d.e. $mx'' + cx' + kx = F(t)$ becomes $x'' + 0.1x' + x = \cos(\omega t)$. Since there is damping in the system, the steady-state solution is the particular solution x_p. Since the nonhomogeneous term is $\cos(\omega t)$, we guess $x_p = A\cos(\omega t) + B\sin(\omega t)$. The homogeneous solution will not contain terms of this form because there is damping, so there is no need to multiply our guess by t. Substituting into the d.e. and solving for A and B, we find that $x_p \approx 10\sin(\omega t) + 0.5\cos(\omega t)$. Thus, the amplitude of the steady-state solution is approximately 10, which is 10 times greater than the amplitude of the forcing term $F(t) = \cos(\omega t)$. This is an example of **resonance**.

4. (Beats) Consider a mass-spring system with mass $m = 1$ kg, spring constant $k = 1$ N/m, damping constant $c = 0$ N·sec/m, and external force $F(t) = 0.44\cos(1.2t)$ N. Suppose the mass starts from rest at the equilibrium position. Find $x(t)$, the position of the mass at time t.

   The model d.e. $mx'' + cx' + kx = F(t)$ becomes $x'' + x = 0.44\cos(1.2t)$. To find x_c, we solve the characteristic equation $r^2 + 1 = 0$, giving $r = \pm i$. Therefore, $x_c = c_1\cos(t) + c_2\sin(t)$. Since the nonhomogeneous term is $0.44\cos(1.2t)$, we guess $x_p = A\cos(1.2t) + B\sin(1.2t)$. Substituting into the d.e. and solving for A and B, we find that $x_p = -\cos(1.2t)$. Thus, $x = c_1\cos(t) + c_2\sin(t) - \cos(1.2t)$. Using the initial conditions $x(0) = 0$, $x'(0) = 0$, we find that $c_1 = 1$ and $c_2 = 0$, so $x = \cos(t) - \cos(1.2t)$. Using the trigonometric identity $\cos(\alpha - \beta) - \cos(\alpha + \beta) = 2\sin(\alpha)\sin(\beta)$ with $\alpha = 1.1t$ and $\beta = 0.1t$, we can rewrite x as $x = [2\sin(0.1t)]\sin(1.1t)$. This can be thought of as a sine function with angular frequency 1.1 and time-dependent amplitude $2\sin(0.1t)$. The amplitude term is periodic, but its frequency is much smaller than that of the "carrier wave" $\sin(1.1t)$. Thus, the amplitude oscillates slowly between 0 and 2. This phenomenon of slowly oscillating amplitude is known as **beats**.