

92.236 Engineering Differential Equations  
Solutions of Homogeneous Linear Equations with Constant Coefficients

Goal: Solve the d.e.

$$ay'' + by' + cy = 0, \tag{1}$$

where  $a$ ,  $b$ , and  $c$  are constants. As discussed in class,  $y = e^{rx}$  is a solution of this d.e. if and only if  $r$  is a root of the quadratic equation

$$ar^2 + br + c = 0. \tag{2}$$

This quadratic equation is known as the *characteristic equation* of the d.e. (1). (Note that the coefficients  $a$ ,  $b$ , and  $c$  in the characteristic equation are the same as the coefficients in the d.e.) We know that there are three possibilities for the roots of a quadratic equation: the equation can have two distinct real roots, a single repeated real root, or two complex roots. The table below gives the form of the solution of the d.e. in each of these three cases.

Characteristic equation $ar^2 + br + c = 0$ has:	General solution of $ay'' + by' + cy = 0$ is:
2 real roots $r_1$ and $r_2$	$y = c_1e^{r_1x} + c_2e^{r_2x}$
1 repeated real root $r$	$y = c_1e^{rx} + c_2xe^{rx}$
Complex roots $\alpha \pm \beta i$	$y = c_1e^{\alpha x} \cos(\beta x) + c_2e^{\alpha x} \sin(\beta x)$

Where did the solutions in the table come from? If the characteristic equation has two real roots, say  $r_1$  and  $r_2$ , then  $y_1 = e^{r_1x}$  and  $y_2 = e^{r_2x}$  are both solutions of the d.e. The solutions  $y_1$  and  $y_2$  are linearly independent because neither is a constant multiple of the other, so the general solution of (1) is

$$y = c_1e^{r_1x} + c_2e^{r_2x} \tag{3}$$

What happens if the characteristic equation has only one real root? For example, consider the d.e.  $y'' + 2y' + y = 0$ . The characteristic equation is  $r^2 + 2r + 1 = 0 \Rightarrow (r + 1)^2 = 0 \Rightarrow r = -1$ . To see how to handle this situation, we rewrite the d.e. as  $D^2y + 2Dy + y = 0$ , where  $Dy$  denotes  $y'$  and  $D^2y$  denotes  $y''$ . Note that the d.e. can be “factored” and rewritten as  $(D + 1)(Dy + y) = 0$ , or  $(D + 1)z = 0$ , where  $z = Dy + y$ . We can solve the first-order d.e.  $(D + 1)z = 0$ , or  $z' + z = 0$ . This equation is both separable and linear, and its solution is  $z = c_1e^{-x}$ . We can now solve the first-order d.e.  $z = Dy + y$  for  $y$ :  $Dy + y = z \Rightarrow y' + y = c_1e^{-x}$ . This is a first-order linear d.e. The integrating factor is  $\rho(x) = e^{\int 1 dx} = e^x$ . Multiplying both sides of the d.e. by the integrating factor, we obtain  $e^x(y' + y) = e^x c_1 e^{-x} = c_1 \Rightarrow \frac{d}{dx}[e^x y] = c_1 \Rightarrow e^x y = \int c_1 dx = c_1 x + c_2 \Rightarrow y = c_1 x e^{-x} + c_2 e^{-x}$ .

We can show by a similar argument that if the characteristic equation has a repeated root  $r$ , then the solution of the d.e. is  $y = c_1e^{rx} + c_2xe^{rx}$ .

The third possibility is that the characteristic equation has complex roots, say  $r_1 = \alpha + \beta i$  and  $r_2 = \alpha - \beta i$ . As discussed in section 3.3 of Edwards & Penney, if we assume that the solution of the d.e. is still  $y = c_1e^{r_1x} + c_2e^{r_2x}$  and use Euler’s formula  $e^{i\theta} = \cos(\theta) + i \sin(\theta)$  to handle the complex exponents, we obtain

$$\begin{aligned} y &= c_1e^{(\alpha+\beta i)x} + c_2e^{(\alpha-\beta i)x} = c_1e^{\alpha x}e^{i\beta x} + c_2e^{\alpha x}e^{-i\beta x} \\ &= c_1e^{\alpha x}[\cos(\beta x) + i \sin(\beta x)] + c_2e^{\alpha x}[\underbrace{\cos(-\beta x)}_{=\cos(\beta x)} + i \underbrace{\sin(-\beta x)}_{=-\sin(\beta x)}] \\ &= (c_1 + c_2)e^{\alpha x} \cos(\beta x) + (ic_1 - ic_2)e^{\alpha x} \sin(\beta x) = d_1e^{\alpha x} \cos(\beta x) + d_2e^{\alpha x} \sin(\beta x) \end{aligned}$$

Of course, the way to check that you really have a solution to the d.e. is to plug it into the d.e. and check that the left side equals the right side.