MATH.2360 Engineering Differential Equations

Solutions of Homogeneous Linear Equations with Constant Coefficients

Goal: Solve the d.e.

$$ay'' + by' + cy = 0, \tag{1}$$

where a, b, and c are constants. As discussed in class, $y = e^{rx}$ is a solution of this d.e. if and only if r is a root of the quadratic equation

$$ar^2 + br + c = 0. \tag{2}$$

This quadratic equation is known as the *characteristic equation* of the d.e. (1). (Note that the coefficients a, b, and c in the characteristic equation are the same as the coefficients in the d.e.) We know that there are three possibilities for the roots of a quadratic equation: the equation can have two distinct real roots, a single repeated real root, or two complex roots. The table below gives the form of the solution of the d.e. in each of these three cases.

Characteristic equation $ar^2 + br + c = 0$ has:	General solution of $ay'' + by' + cy = 0$ is:
2 real roots r_1 and r_2	$y = c_1e^{r_1x} + c_2e^{r_2x}$
1 repeated real root r	$y = c_1e^{rx} + c_2xe^{rx}$
Complex roots $\alpha \pm \beta i$	$y = c_1e^{\alpha x}\cos(\beta x) + c_2e^{\alpha x}\sin(\beta x)$

Where did the solutions in the table come from? If the characteristic equation has two real roots, say r_1 and r_2, then $y_1 = e^{r_1x}$ and $y_2 = e^{r_2x}$ are both solutions of the d.e. The solutions y_1 and y_2 are linearly independent because neither is a constant multiple of the other, so the general solution of (1) is

$$y = c_1e^{r_1x} + c_2e^{r_2x} \tag{3}$$

What happens if the characteristic equation has only one real root? For example, consider the d.e. $y'' + 2y' + y = 0$. The characteristic equation is $r^2 + 2r + 1 = 0 \Rightarrow (r + 1)^2 = 0 \Rightarrow r = -1$. To see how to handle this situation, we rewrite the d.e. as $D^2y + 2Dy + y = 0$, where Dy denotes y' and D^2y denotes y''. Note that the d.e. can be "factored" and rewritten as $(D + 1)(Dy + y) = 0$, or $(D + 1)z = 0$, where $z = Dy+y$. We can solve the first-order d.e. $(D + 1)z = 0$, or $z'+z = 0$. This equation is both separable and linear, and its solution is $z = c_1e^{-x}$. We can now solve the first-order d.e. $z = Dy+y$ for y: $Dy+y = z \Rightarrow y'+y = c_1e^{-x}$. This is a first-order linear d.e. The integrating factor is $\rho(x) = e^{\int 1\,dx} = e^x$. Multiplying both sides of the d.e. by the integrating factor, we obtain $e^x(y' + y) = e^xc_1e^{-x} = c_1 \Rightarrow \frac{d}{dx}[e^xy] = c_1 \Rightarrow e^xy = \int c_1\,dx = c_1x + c_2 \Rightarrow y = c_1xe^{-x} + c_2e^{-x}$. We can show by a similar argument that if the characteristic equation has a repeated root r, then the solution of the d.e. is $y = c_1e^{rx} + c_2xe^{rx}$.

The third possibility is that the characteristic equation has complex roots, say $r_1 = \alpha + \beta i$ and $r_2 = \alpha - \beta i$. As discussed in section 3.3 of Edwards & Penney, if we assume that the solution of the d.e. is still $y = c_1e^{r_1x} + c_2e^{r_2x}$ and use Euler's formula $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ to handle the complex exponents, we obtain

$$\begin{aligned} y &= c_1e^{(\alpha+\beta i)x} + c_2e^{(\alpha-\beta i)x} = c_1e^{\alpha x}e^{i\beta x} + c_2e^{\alpha x}e^{-i\beta x} \\ &= c_1e^{\alpha x}[\cos(\beta x) + i\sin(\beta x)] + c_2e^{\alpha x}[\underbrace{\cos(-\beta x)}_{=\cos(\beta x)} + i\underbrace{\sin(-\beta x)}_{=-\sin(\beta x)}] \\ &= (c_1 + c_2)e^{\alpha x}\cos(\beta x) + (ic_1 - ic_2)e^{\alpha x}\sin(\beta x) = d_1e^{\alpha x}\cos(\beta x) + d_2e^{\alpha x}\sin(\beta x) \end{aligned}$$

Of course, the way to check that you really have a solution to the d.e. is to plug it into the d.e. and check that the left side equals the right side.