Engineering Differential Equations

Example using Laplace Transform to Solve a DE with Discontinuous Forcing Term

Problem: Solve the IVP $x'' + 4x = F(t),\ x(0) = 0,\ x'(0) = 2$, where

$$F(t) = \begin{cases} 0 & \text{if } t < 2\pi \\ 6\sin(t) & \text{if } t \geq 2\pi \end{cases}$$

Note that $F(t) = 6u\,(t - 2\pi)\sin(t - 2\pi)$.

Step 1. Take the transform of both sides of the d.e:

$$\mathcal{L}\left\{x'' + 4x\right\} = \mathcal{L}\left\{6u\,(t - 2\pi)\sin(t - 2\pi)\right\}$$

Step 2. Solve for $\mathcal{L}\left\{x\right\}$:

$$\mathcal{L}\left\{x''\right\} + 4\mathcal{L}\left\{x\right\} = 6\mathcal{L}\left\{u(t - 2\pi)\sin(t - 2\pi)\right\} \Rightarrow$$
$$s^2\mathcal{L}\left\{x\right\} - sx(0) - x'(0) + 4\mathcal{L}\left\{x\right\} = 6e^{-2\pi s}\mathcal{L}\left\{\sin(t)\right\} \Rightarrow$$
$$s^2\mathcal{L}\left\{x\right\} - s\cdot 0 - 2 + 4\mathcal{L}\left\{x\right\} = 6e^{-2\pi s}\frac{1}{s^2+1} \Rightarrow$$
$$\left(s^2 + 4\right)\mathcal{L}\left\{x\right\} = 2 + 6e^{-2\pi s}\frac{1}{s^2+1} \Rightarrow$$
$$\mathcal{L}\left\{x\right\} = \frac{2}{s^2+4} + e^{-2\pi s}\frac{6}{\left(s^2+1\right)\left(s^2+4\right)}$$

Step 3. Solve for x:

$$x = \mathcal{L}^{-1}\left\{\frac{2}{s^2+4} + e^{-2\pi s}\frac{6}{\left(s^2+1\right)\left(s^2+4\right)}\right\} \Rightarrow$$
$$x = \mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} + \mathcal{L}^{-1}\left\{e^{-2\pi s}\frac{6}{\left(s^2+1\right)\left(s^2+4\right)}\right\} \Rightarrow$$
$$x = \sin(2t) + u(t - 2\pi)f(t - 2\pi),$$
where $f(t) = \mathcal{L}^{-1}\left\{\frac{6}{\left(s^2+1\right)\left(s^2+4\right)}\right\} = \mathcal{L}^{-1}\left\{\frac{2}{\left(s^2+1\right)} - \frac{2}{\left(s^2+4\right)}\right\} = 2\sin(t) - \sin(2t) \Rightarrow$

$$x = \sin(2t) + u(t-2\pi)\left[2\sin\left(t - 2\pi\right) - \sin\left(2(t - 2\pi)\right)\right] \Rightarrow \boxed{x = \sin(2t) + u(t - 2\pi)\left[2\sin(t) - \sin(2t)\right]}.$$