**92.236 Engineering Differential Equations Final Exam Solutions**
**Fall 2011**

## Problem 1. (10 pts.)

Solve the following initial value problem: $y' = \dfrac{1}{3y^2\,(2x+1)}$ with $y(0) = 1$. Note: $y'$ means $dy/dx$.

This is a separable d.e. $\boxed{\text{2 pts.}}$ $\dfrac{dy}{dx} = \dfrac{1}{3y^2\,(2x+1)} \Rightarrow 3y^2\ dy = \dfrac{dx}{2x+1}$ $\boxed{\text{2 pts.}}$

$\Rightarrow \displaystyle\int 3y^2\ dy = \int \frac{dx}{2x+1} \Rightarrow y^3 = \frac{1}{2}\ln(2x+1) + c$ $\boxed{\text{5 pts.}}$

$y(0) = 1 \Rightarrow 1^3 = \dfrac{1}{2}\ln(2(0)+1) + c \Rightarrow c = 1$ $\boxed{\text{1 pt.}}$

Therefore, $y^3 = \dfrac{1}{2}\ln(2x+1) + 1 \Rightarrow \boxed{\boxed{y = \left[\dfrac{1}{2}\ln(2x+1) + 1\right]^{1/3}.}}$

## Problem 2. (10 pts.)

Solve the following initial value problem: $y' = \dfrac{2x+y}{x}$, $y(1) = 2$.

Express your solution $y$ explicitly in terms of $x$. In other words, write your answer in the form $y =$ something. Note: $y'$ means $dy/dx$.

This d.e. is both linear (because $y$ and $y'$ appear only to the first power) and homogeneous (because $dy/dx$ equals a rational function in which each term has the same degree, namely 1). $\boxed{\text{2 pts.}}$

To treat the d.e. as linear, first write the d.e. in standard form: $y' = \dfrac{2x+y}{x} = \dfrac{2x}{x} + \dfrac{y}{x} = 2 + \dfrac{y}{x} \Rightarrow$
$y' - \left(\dfrac{1}{x}\right) y = 2.$ $\boxed{\text{1 pt.}}$

The integrating factor is $\rho(x) = e^{\int -1/x\ dx} = e^{-\ln(x)} = x^{-1}$. $\boxed{\text{2 pts.}}$
Multiplying both sides of the standard form of the d.e. by the integrating factor, we have
$x^{-1}\left[y' - \left(\dfrac{1}{x}\right) y\right] = 2x^{-1} \Rightarrow \dfrac{d}{dx}\left[x^{-1}y\right] = 2x^{-1} \Rightarrow x^{-1}y = \displaystyle\int 2x^{-1}\ dx = 2\ln(x) + c.$ $\boxed{\text{3 pts.}}$

$y(1) = 2 \Rightarrow 1^{-1}(2) = 2\ln(1) + c \Rightarrow c = 2.$ $\boxed{\text{1 pt.}}$

Therefore, $x^{-1}y = 2\ln(x) + 2 \Rightarrow \boxed{\boxed{y = 2x\ln(x) + 2x.}}$ $\boxed{\text{1 pt.}}$

If you regard the d.e. as a homogeneous equation, let $v = y/x$, replace $y'$ by $v + xv'$, and replace $y$ by $xv$: $y' = \dfrac{2x+y}{x} \Rightarrow v + xv' = \dfrac{2x+xv}{x} = \dfrac{x(2+v)}{x} = 2 + v \Rightarrow xv' = 2$ $\boxed{\text{3 pts.}}$

This d.e. is separable: $x\dfrac{dv}{dx} = 2 \Rightarrow dv = \dfrac{2}{x}\ dx \Rightarrow \displaystyle\int\ dv = \int \frac{2}{x}\ dx \Rightarrow v = 2\ln(x) + c \Rightarrow y/x = 2\ln(x) + c$
$\boxed{\text{3 pts.}}$

$y(1) = 2 \Rightarrow 2/1 = 2\ln(1) + c \Rightarrow c = 2.$ $\boxed{\text{1 pt.}}$

Therefore, $y/x = 2\ln(x) + 2 \Rightarrow \boxed{\boxed{y = 2x\ln(x) + 2x.}}$ $\boxed{\text{1 pt.}}$

**Problem 3. (15 points)**

Let $t$ denote time (in days) and let $P$ denote the size of a mosquito population (in grams) at time $t$. Suppose the daily birth rate per gram is $\beta = 6 - 2P$, and suppose the daily death rate per gram is $\delta = 2$. (The units of $\beta$ and $\delta$ are (gram/day)/gram.)

a. (6 pts.) Write down the differential equation modeling this problem ($\frac{dP}{dt} =$ something).

$\frac{dP}{dt} =$ rate in - rate out $= \beta P - \delta P$ $\boxed{\text{3 pts.}}$ so $\frac{dP}{dt} = (6-2P)P - 2P \Rightarrow$

$\boxed{\boxed{\frac{dP}{dt} = 4P - 2P^2}}$ $\boxed{\text{3 pts.}}$

b. (6 pts.) Draw the phase line for the d.e. from part a.

$4P - 2P^2 = 0 \Rightarrow 2P(2-P) = 0 \Rightarrow$ the critical points are 0 and 2 $\boxed{\text{1 pt.}}$

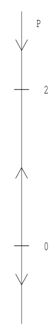
The two critical points divide the phase line into 3 intervals: $P > 2,\ 0 < P < 2,$ and $P < 0$.

$\left.\frac{dx}{dt}\right|_{P=3} = 2(3)(2-3) < 0$, so the direction arrow points down for $P > 2$.

$\left.\frac{dx}{dt}\right|_{P=1} = 2(1)(2-1) > 0$, so the direction arrow points up for $0 < P < 2$.

$\left.\frac{dx}{dt}\right|_{P=-1} = 2(-1)(2-(-1)) < 0$, so the arrow points down for $P < 0$.

$\boxed{\text{3 pts.}}$




$\boxed{\text{2 pts.}}$

c. (3 pts.) Suppose $P(0) = 1$. Use your phase line to find the limiting value of $P(t)$ as $t$ increases.

Since 1 lies in the interval $0 < P < 2$, we can see from the phase line that $\boxed{\boxed{P(t) \to 2}}$ as $t$ increases. $\boxed{\text{3 pts.}}$

**Problem 4. (10 pts.)** Find the general solution to each of the following differential equations.

a. (4 points) $y'' + 2y' + 5y = 0$

The characteristic equation is $r^2 + 2r + 5 = 0 \Rightarrow$

$$r = \frac{-2 \pm \sqrt{2^2 - 4(1)(5)}}{2(1)} = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i \ \boxed{\text{2 pts.}}$$

Therefore, $\boxed{\boxed{y = c_1 e^{-x}\cos(2x) + c_2 e^{-x}\sin(2x)}}$. $\boxed{\text{2 pts.}}$

b. (6 points) $y^{(4)} + 5y^{(3)} + 6y'' = 0$

The characteristic equation is $r^4 + 5r^3 + 6r^2 = 0 \Rightarrow r^2\left(r^2 + 5r + 6\right) = 0 \Rightarrow r^2(r+3)(r+2) = 0$.

Thus, the roots of the characteristic equation are 0 (double root), −3, and −2. $\boxed{\text{2 pts.}}$

Therefore, $y = c_1e^{0x} + c_2xe^{0x} + c_3e^{-3x} + c_4e^{-2x}$ or $\boxed{\boxed{y = c_1 + c_2x + c_3e^{-3x} + c_4e^{-2x}}}$. $\boxed{\text{4 pts.}}$

**Problem 5. (15 points)**

Solve the following initial value problem:

$$y'' - y' - 2y = 4x - 8e^{3x} \text{ with } y(0) = 2 \text{ and } y'(0) = -8.$$

Step 1. Find $y_c$ by solving the homogeneous d.e. $y'' - y' - 2y$.

Characteristic equation: $r^2 - r - 2 = 0 \Rightarrow (r+1)(r-2) = 0 \Rightarrow r = -1$or $r = 2$.

Therefore, $y_c = c_1e^{-x} + c_2e^{2x}$. $\boxed{\text{3 pts.}}$

Step 2. Find $y_p$ using **either** the Method of Undetermined Coefficients **or** the Method of Variation of Parameters.

Method 1. Undetermined Coefficients.

Since the nonhomogeneous term in the d.e. $(4x - 8e^{3x})$ is the sum of a polynomial of degree 1 and an exponential function, we guess that $y_p$ is the sum of a polynomial of degree 1 and an exponential function: $y_p = Ax + B + Ce^{3x}$. No term in this guess duplicates a term in $y_c$, so there is no need to modify the guess. $\boxed{\text{3 pts.}}$ $y = Ax + B + Ce^{3x} \Rightarrow y' = A + 3Ce^{3x} \Rightarrow y'' = 9Ce^{3x}$.

Therefore, the left side of the d.e. is
$y'' - y' - 2y = 9Ce^{3x} - \left[A + 3Ce^{3x}\right] - 2\left[Ax + B + Ce^{3x}\right] = 4Ce^{3x} - 2Ax - A - 2B$. We want this to equal the nonhomogeneous term $4x - 8e^{3x}$, so $-2A = 4,\ -A - 2B = 0$, and $4C = -8 \Rightarrow A = -2,\ B = 1$, and $C = -2$. Therefore, $y_p = -2x + 1 - 2e^{3x}$. $\boxed{\text{6 pts.}}$

Method 2. Variation of Parameters.

From $y_c$ we obtain two independent solutions of the homogeneous d.e: $y_1 = e^{-x}$ and $y_2 = e^{2x}$. The Wronskian is given by $W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x} & e^{2x} \\ -e^{-x} & 2e^{2x} \end{vmatrix}$

$= e^{-x}\left(2e^{2x}\right) - (-e^{-x})(e^{2x}) = 2e^{x} + e^{x} = 3e^{x}$. $\boxed{\text{1 pt.}}$

$u_1 = \int \frac{-y_2 f(x)}{W(x)}\,dx = -\int \frac{e^{2x}\left(4x - 8e^{3x}\right)}{3e^{x}}\,dx = -\frac{4}{3}\int xe^{x}\,dx + \frac{8}{3}\int e^{4x}\,dx = -\frac{4}{3}(x-1)e^{x} + \frac{2}{3}e^{4x}$

using entry 46 from the Table of Integrals. $\boxed{\text{3 pts.}}$

$u_2 = \int \frac{y_1 f(x)}{W(x)}\,dx = \int \frac{e^{-x}\left(4x - 8e^{3x}\right)}{3e^{x}}\,dx = \frac{4}{3}\int xe^{-2x}\,dx - \frac{8}{3}\int e^{x}\,dx = \frac{1}{3}\left(-2x - 1\right)e^{-2x} - \frac{8}{3}e^{x}$

using entry 46 from the Table of Integrals with $u = -2x$. $\boxed{\text{3 pts.}}$

Therefore, $y_p = u_1y_1 + u_2y_2 = \left[-\frac{4}{3}(x-1)e^{x} + \frac{2}{3}e^{4x}\right]e^{-x} + \left[\frac{1}{3}\left(-2x-1\right)e^{-2x} - \frac{8}{3}e^{x}\right]e^{2x}$

$= -\frac{4}{3}(x-1) + \frac{2}{3}e^{3x} + \frac{1}{3}\left(-2x-1\right) - \frac{8}{3}e^{3x} = -2x + 1 - 2e^{3x}$ $\boxed{\text{2 pts.}}$

Step 3. $y = y_c + y_p$, so $y = c_1e^{-x} + c_2e^{2x} - 2x + 1 - 2e^{3x}$. $\boxed{\text{1 pt.}}$

Step 4. Use the initial conditions to find $c_1$ and $c_2$.

$y = c_1e^{-x} + c_2e^{2x} - 2x + 1 - 2e^{3x} \Rightarrow y' = -c_1e^{-x} + 2c_2e^{2x} - 2 - 6e^{3x}$

$y(0) = 2 \Rightarrow 2 = c_1 e^0 + c_2 e^0 - 2(0) + 1 - 2e^0 = c_1 + c_2 - 1 \Rightarrow c_1 + c_2 = 3$
$y'(0) = -8 \Rightarrow -8 = -c_1 e^0 + 2c_2 e^0 - 2 - 6e^0 = -c_1 + 2c_2 - 8 \Rightarrow -c_1 + 2c_2 = 0$
$c_1 + c_2 = 3, \ -c_1 + 2c_2 = 0 \Rightarrow c_1 = 2, \ c_2 = 1$ $\boxed{\text{2 pts.}}$

Therefore, $\boxed{\boxed{y = 2e^{-x} + e^{2x} - 2x + 1 - 2e^{3x}.}}$

**Problem 6. (15 points)**

Consider a damped, forced mass/spring system. Let $t$ denote time (in seconds) and let $x(t)$ denote the position (in meters) of the mass at time $t$, with $x = 0$ corresponding to the equilibrium position. Suppose the mass $m = 2$ kg, the damping constant $c = 8$ N·s/m, the spring constant $k = 6$ N/m, and the external force is $F_e(t) = 240\cos(3t)$.

a. (13 pts.) Find the steady-state (steady periodic) solution $x_{sp}$.

The d.e. modeling this system is $mx'' + cx' + kx = F(t)$, or $2x'' + 8x' + 6x = 240\cos(3t)$, which gives $x'' + 4x' + 3x = 120\cos(3t)$. $\boxed{\text{2 pts.}}$

The steady-state (steady periodic) solution $x_{sp}$ is the particular solution $x_p$. $\boxed{\text{2 pts.}}$

You can find $x_p$ using either the Method of Undetermined Coefficients the Method of Variation of Parameters. Here we use the Method of Undetermined Coefficients to save the work of finding $x_c$.

Since the nonhomogeneous term in the d.e. ($120\cos(3t)$) is a cosine, we guess that $x_p$ is the sum of a cosine and sine with the same frequency: $x_p = A\cos(3t) + B\sin(3t)$. The complementary solution $x_c$ will contain decaying exponential terms because of the damping term in the d.e., so we know that no term in our guess for $x_p$ duplicates a term in $x_c$. Therefore, there is no need to modify the guess. $\boxed{\text{4 pts.}}$

$x = A\cos(3t) + B\sin(3t) \Rightarrow x' = -3A\sin(3t) + 3B\cos(3t) \Rightarrow x'' = -9A\cos(3t) - 9B\sin(3t)$.

Therefore, the left side of the d.e. is

$x'' + 4x' + 3x = -9A\cos(3t) - 9B\sin(3t) + 4\left[-3A\sin(3t) + 3B\cos(3t)\right] + 3\left[A\cos(3t) + B\sin(3t)\right] = [-6A + 12B]\cos(3t) + [-12A - 6B]\sin(3t)$. We want this to equal the nonhomogeneous term $120\cos(3t)$, so $-6A + 12B = 120$ and $-12A - 6B = 0 \Rightarrow A = -4$ and $B = 8$. Therefore,

$\boxed{\boxed{x_{sp} = -4\cos(3t) + 8\sin(3t)}}$ $\boxed{\text{5 pts.}}$

b. (2 pts.) Express your answer to part a in the form $x_{sp}(t) = C\cos(\omega t - \alpha)$

$C = \sqrt{A^2 + B^2} = \sqrt{(-4)^2 + 8^2} = \sqrt{80} = 4\sqrt{5}$ $\boxed{\text{1 pt.}}$

Since $A < 0, \ \alpha = \pi + \tan^{-1}(B/A) = \pi + \tan^{-1}(-2) \approx 2.0344$ $\boxed{\text{1 pt.}}$

**Problem 7. (10 points)**

a. (3 pts.) Find $\mathcal{L}\left\{\sqrt{t} + \sin(2t)\right\}$

Using the Laplace Transform table entries for $t^a$ and $\sin(kt)$, we find that

$$\boxed{\boxed{\mathcal{L}\left\{\sqrt{t} + \sin(2t)\right\} = \frac{\Gamma(3/2)}{s^{3/2}} + \frac{2}{s^2 + 4}}} \ \boxed{\text{3 pts.}}$$

b. (7 pts.) Find $\mathcal{L}^{-1}\left\{\dfrac{8}{s^2-6s+25}\right\}$.

If we were to try a partial fraction decomposition, we would first have to factor the denominator. However, the roots of the denominator are complex, so we cannot factor the denominator. This means we have to complete the square: $s^2-6s+25=(s-3)^2+16$ $\boxed{\text{4 pts.}}$ Therefore, $\mathcal{L}^{-1}\left\{\dfrac{8}{s^2-6s+25}\right\}=\mathcal{L}^{-1}\left\{\dfrac{8}{(s-3)^2+16}\right\}=2\mathcal{L}^{-1}\left\{\dfrac{4}{(s-3)^2+16}\right\}=\boxed{\boxed{2e^{3t}\sin(4t)}}$ using the Laplace Transform table entry for $e^{at}\sin(kt)$ $\boxed{\text{3 pts.}}$

**Problem 8. (15 points)**

*Use the Laplace Transform* to solve the following initial value problem:

$$x''+4x=10e^t \quad \text{with } x(0)=0 \text{ and } x'(0)=2.$$

**Solutions to this IVP not using the Laplace transform method will not receive any credit.** Primes denote derivatives with respect to $t$: $x'=dx/dt$ and $x''=d^2x/dt^2$.

$x''+4x=10e^t \Rightarrow \mathcal{L}\{x''+4x\}=\mathcal{L}\left\{10e^t\right\} \Rightarrow \mathcal{L}\{x''\}+4\mathcal{L}\{x\}=10\mathcal{L}\left\{e^t\right\}$ $\boxed{\text{3 pts.}}$

$\Rightarrow s^2\mathcal{L}\{x\}-sx(0)-x'(0)+4\mathcal{L}\{x\}=10\left(\dfrac{1}{s-1}\right)$ $\boxed{\text{3 pts.}}$

$s^2\mathcal{L}\{x\}-s\cdot 0-2+4\mathcal{L}\{x\}=10\left(\dfrac{1}{s-1}\right) \Rightarrow \left(s^2+4\right)\mathcal{L}\{x\}=10\left(\dfrac{1}{s-1}\right)+2=\dfrac{2s+8}{s-1} \Rightarrow$

$\mathcal{L}\{x\}=\dfrac{2s+8}{(s-1)\left(s^2+4\right)}$ $\boxed{\text{1 pt.}}$ $\Rightarrow x=\mathcal{L}^{-1}\left\{\dfrac{2s+8}{(s-1)\left(s^2+4\right)}\right\}$.

Use a partial fraction decomposition. $\dfrac{2s+8}{(s-1)\left(s^2+4\right)}=\dfrac{A}{s-1}+\dfrac{Bs+C}{s^2+4}$

$\Rightarrow 2s+8=\left[\dfrac{A}{s-1}+\dfrac{Bs+C}{s^2+4}\right](s-1)\left(s^2+4\right)$

$\Rightarrow 2s+8=A\left(s^2+4\right)+(Bs+C)(s-1)$

$\Rightarrow 2s+8=As^2+4A+Bs^2-Bs+Cs-C=(A+B)s^2+(C-B)s+4A-C$

$\Rightarrow A+B=0,\ C-B=2,\ 4A-C=8 \Rightarrow A=2,\ B=-2,\ C=0$

$\Rightarrow x=\mathcal{L}^{-1}\left\{\dfrac{2s+8}{(s-1)\left(s^2+4\right)}\right\}=\mathcal{L}^{-1}\left\{\dfrac{2}{s-1}+\dfrac{-2s}{s^2+4}\right\}=2\mathcal{L}^{-1}\left\{\dfrac{1}{s-1}\right\}-2\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+4}\right\}$

Using the table entries for $e^{at}$ and $\cos(kt)$, we get $\boxed{\boxed{x=2e^t-2\cos(2t)}}$. $\boxed{\text{8 pts.}}$