92.236 Engineering Differential Equations   Final Exam Solutions
Fall 2012

Problem 1. (10 pts.)

Solve the following initial value problem: $\dfrac{dy}{dx} = \dfrac{3x^2}{3y^2+1}$ with $y(1) = 2$.

This is a separable d.e. $\boxed{\text{2 pts.}}$ $\dfrac{dy}{dx} = \dfrac{3x^2}{3y^2+1} \Rightarrow \left(3y^2+1\right)\ dy = 3x^2\ dx.$ $\boxed{\text{2 pts.}}$

$\Rightarrow \displaystyle\int \left(3y^2+1\right)\ dy = \int 3x^2\ dx \Rightarrow y^3 + y = x^3 + c$ $\boxed{\text{5 pts.}}$

$y(1) = 2 \Rightarrow 2^3 + 2 = 1^3 + c \Rightarrow c = 9.$ $\boxed{\text{1 pt.}}$ Therefore, $\boxed{\boxed{y^3 + y = x^3 + 9.}}$

Problem 2. (10 pts.)

Solve the following initial value problem: $x^2\dfrac{dy}{dx} = xy + 2, \quad y(1) = 2.$

Express your solution y explicitly in terms of x. In other words, write your answer in the form $y =$ something.

This is a linear d.e. because y and dy/dx appear only to the first power. $\boxed{\text{2 pts.}}$

First write the d.e. in standard form: $\dfrac{dy}{dx} - \left(\dfrac{1}{x}\right) y = \dfrac{2}{x^2}.$ $\boxed{\text{1 pt.}}$

The integrating factor is $\rho(x) = e^{\int -1/x\ dx} = e^{-\ln(x)} = x^{-1}.$ $\boxed{\text{2 pts.}}$

Multiplying both sides of the standard form of the d.e. by the integrating factor, we have

$x^{-1}\left[\dfrac{dy}{dx} - \left(\dfrac{1}{x}\right) y\right] = x^{-1}\left(2x^{-2}\right)$ $\boxed{\text{1 pt.}}$ $\Rightarrow x^{-1}\dfrac{dy}{dx} - x^{-2}y = 2x^{-3} \Rightarrow \dfrac{d}{dx}\left[x^{-1}y\right] = 2x^{-3} \Rightarrow$

$x^{-1}y = \displaystyle\int 2x^{-3}\ dx = -x^{-2} + c.$ $\boxed{\text{2 pts.}}$ $\quad y(1) = 2 \Rightarrow 1^{-1}(2) = -1^{-2} + c \Rightarrow c = 3.$ $\boxed{\text{1 pt.}}$

Therefore, $x^{-1}y = -x^{-2} + 3 \Rightarrow \boxed{\boxed{y = 3x - x^{-1}.}}$ $\boxed{\text{1 pt.}}$

Problem 3. (15 points)

A cup of coffee at temperature 120° F is brought into a room where the temperature is 70° F. After 10 minutes the coffee temperature is 110° F. When will the coffee temperature reach 100° F?

Recall that the de modeling heating/cooling problems is $\dfrac{dT}{dt} = -k\left(T - A\right).$

As we found in class, $\dfrac{dT}{dt} = -k\left(T-A\right), T(0) = T_0 \Rightarrow T = A + \left(T_0 - A\right)e^{-kt}$ $\boxed{\text{5 pts.}}$

In this problem, $A = 70$ and $T_0 = 120$, so $T = 70 + (120 - 70)\,e^{-kt} = 70 + 50e^{-kt}$.

$T(10) = 110 \Rightarrow 110 = 70 + 50e^{-k(10)} \Rightarrow 40 = 50e^{-10k} \Rightarrow \underbrace{\dfrac{40}{50}}_{0.8} = e^{-10k} \Rightarrow \ln(0.8) = \ln\left(e^{-10k}\right) = -10k \Rightarrow$

$k = -\ln(0.8)/10$ $\boxed{\text{5 pts.}}$

Let t_{100} denote the time at which the coffee temperature reaches 100 degrees.

$T(t_{100}) = 100 \Rightarrow 100 = 70 + 50e^{-k(t_{100})} \Rightarrow 30 = 50e^{-kt_{100}} \Rightarrow \underbrace{\frac{30}{50}}_{0.6} = e^{-kt_{100}} \Rightarrow \ln(0.6) = \ln\left(e^{-kt_{100}}\right) = -kt_{100} \Rightarrow$

$t_{100} = -\ln(0.6)/k = \boxed{\boxed{\dfrac{10\ln(0.6)}{\ln(0.8)} \approx 23 \text{ minutes.}}}$ $\boxed{\text{5 pts.}}$

Problem 4. (10 points) Find the general solution to each of the following differential equations:

a. (4 points) $y'' + 2y' + y = 0$

The characteristic equation is $r^2 + 2r + 1 = 0 \Rightarrow$
$(r+1)^2 = 0 \Rightarrow r = -1$ (double root). $\boxed{\text{2 pts.}}$

Therefore, $\boxed{\boxed{y = c_1 e^{-x} + c_2 x e^{-x}}}$. $\boxed{\text{2 pts.}}$

b. (6 points) $y^{(4)} + y'' = 0$

The characteristic equation is $r^4 + r^2 = 0 \Rightarrow r^2\left(r^2+1\right) = 0.$

Thus, the roots of the characteristic equation are 0 (double root) and $\pm i = 0 \pm i$. $\boxed{\text{2 pts.}}$
Therefore, $y = c_1 e^{0x} + c_2 x e^{0x} + c_3 e^{0x}\sin(1x) + c_4 e^{0x}\sin(1x)$ or
$\boxed{\boxed{y = c_1 + c_2 x + c_3\cos(x) + c_4\sin(x)}}$. $\boxed{\text{4 pts.}}$

Problem 5. (15 points)

Solve the following initial value problem:

$$y'' - y' - 2y = 4x + 12e^{3x} \quad \text{with } y(0) = 9 \text{ and } y'(0) = 2.$$

Note: $y' = dy/dx$ and $y'' = d^2y/dx^2$

Step 1. Find y_c by solving the homogeneous d.e. $y'' - y' - 2y = 0$.
Characteristic equation: $r^2 - r - 2 = 0 \Rightarrow (r+1)(r-2) = 0 \Rightarrow r = -1$ or $r = 2$.
Therefore, $y_c = c_1 e^{-x} + c_2 e^{2x}$. $\boxed{\text{3 pts.}}$
Step 2. Find y_p using **either** the Method of Undetermined Coefficients **or** the Method of Variation of Parameters.

Method 1. Undetermined Coefficients.

Since the nonhomogeneous term in the d.e. $(4x + 12e^{3x})$ is the sum of a polynomial of degree 1 and an exponential function, we should guess that y_p is the sum of a polynomial of degree 1 and an exponential function: $y_p = Ax + B + Ce^{3x}$. No term in this guess duplicates a term in y_c, so there is no need to modify the guess. $\boxed{\text{3 pts.}}$ $y = Ax + B + Ce^{3x} \Rightarrow y' = A + 3Ce^{3x} \Rightarrow y'' = 9Ce^{3x}$.

Therefore, the left side of the d.e. is
$y'' - y' - 2y = 9Ce^{3x} - \left[A + 3Ce^{3x}\right] - 2\left[Ax + B + Ce^{3x}\right] = 4Ce^{3x} - 2Ax + (-A - 2B)$. We want this to equal the nonhomogeneous term $4x + 12e^{3x}$, so
$-2A = 4,\ -A - 2B = 0,$ and $4C = 12 \Rightarrow A = -2,\ B = 1,$ and $C = 3$.
Therefore, $y_p = 3e^{3x} - 2x + 1$. $\boxed{\text{6 pts.}}$

Method 2. Variation of Parameters.

From y_c we obtain two independent solutions of the homogeneous d.e: $y_1 = e^{-x}$ and
$y_2 = e^{2x}$. The Wronskian is given by $W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x} & e^{2x} \\ -e^{-x} & 2e^{2x} \end{vmatrix}$

$= e^{-x}\left(2e^{2x}\right) - (-e^{-x})e^{2x} = 3e^{x}.$ $\boxed{\text{1 pt.}}$

$$u_1 = \int \frac{-y_2 f(x)}{W(x)}\,dx = -\int \frac{e^{2x}\left(4x+12e^{3x}\right)}{3e^{x}}\,dx = -\int \left[\frac{4}{3}xe^{x} + 4e^{4x}\right]\,dx$$

For the first term in the integrand, use formula 46 from the Table of Integrals. For the second term in the integrand, use substitution with $u = 4x$.

$$u_1 = -\int \left[\frac{4}{3}xe^{x} + 4e^{4x}\right]\,dx = -\frac{4}{3}(x-1)e^{x} - e^{4x} \quad \boxed{\text{3 pts.}}$$

$$u_2 = \int \frac{y_1 f(x)}{W(x)}\,dx = \int \frac{e^{-x}\left(4x+12e^{3x}\right)}{3e^{x}}\,dx = \int \left[\frac{4}{3}xe^{-2x} + 4e^{x}\right]\,dx$$

For the first integral, substitute $u = -2x$ and then use formula 46 from the Table of Integrals.

$$u_2 = \int \left[\frac{4}{3}xe^{-2x} + 4e^{x}\right]\,dx = \frac{1}{3}(-2x-1)e^{-2x} + 4e^{x} \quad \boxed{\text{3 pts.}}$$

Therefore,

$$y_p = u_1y_1 + u_2y_2 = \left[-\frac{4}{3}(x-1)e^{x} - e^{4x}\right]e^{-x} + \left[\frac{1}{3}(-2x-1)e^{-2x} + 4e^{x}\right]e^{2x}$$

$= -2x + 1 + 3e^{3x}$ $\boxed{\text{2 pts.}}$

Step 3. $y = y_c + y_p$, so $y = c_1e^{-x} + c_2e^{2x} + 3e^{3x} - 2x + 1.$ $\boxed{\text{1 pt.}}$

Step 4. Use the initial conditions to find c_1 and c_2.

$y = c_1e^{-x} + c_2e^{2x} + 3e^{3x} - 2x + 1 \Rightarrow y' = -c_1e^{-x} + 2c_2e^{2x} + 9e^{3x} - 2$

$y(0) = 9 \Rightarrow 9 = c_1e^{0} + c_2e^{0} + 3e^{0} - 2(0) + 1 = c_1 + c_2 + 4 \Rightarrow c_1 + c_2 = 5$

$y'(0) = 2 \Rightarrow 2 = -c_1e^{0} + 2c_2e^{0} + 9e^{0} - 2 = -c_1 + 2c_2 + 7 \Rightarrow -c_1 + 2c_2 = -5$

$c_1 + c_2 = 5$ and $-c_1 + 2c_2 = -5 \Rightarrow c_1 = 5,\ c_2 = 0$ $\boxed{\text{2 pts.}}$

Therefore, $\boxed{\boxed{y = 5e^{-x} + 3e^{3x} - 2x + 1.}}$

Problem 6. (15 points)

Find the position function $x(t)$ for an unforced, damped mass-spring system with mass $m = 1$ kg, damping coefficient $c = 4$ Ns/m, and spring constant $k = 5$ N/m. Take $x(0) = 2$ m and $x'(0) = 0$ m/s.

The d.e. modeling this system is $mx'' + cx' + kx = F_e(t)$, or $x'' + 4x' + 5x = 0.$ $\boxed{\text{4 pts.}}$

Characteristic equation: $r^2 + 4r + 5 = 0 \Rightarrow r = \dfrac{-4 \pm \sqrt{4^2 - 4(1)(5)}}{2(1)} = \dfrac{-4 \pm 2i}{2} = -2 \pm i \Rightarrow$ $\boxed{\text{4 pts.}}$

$x = c_1e^{-2t}\cos(t) + c_2e^{-2t}\sin(t).$ $\boxed{\text{5 pts.}}$

$$x = c_1e^{-2t}\cos(t) + c_2e^{-2t}\sin(t) \Rightarrow x' = c_1\left[-2e^{-2t}\cos(t) - e^{-2t}\sin(t)\right] + c_2\left[-2e^{-2t}\sin(t) + e^{-2t}\cos(t)\right]$$

$x(0) = 2 \Rightarrow 2 = c_1e^{0}\cos(0) + c_2e^{0}\sin(0) = c_1$

$x'(0) = 0 \Rightarrow 0 = c_1\left[-2e^{0}\cos(0) - e^{0}\sin(0)\right] + c_2\left[-2e^{0}\sin(0) + e^{0}\cos(0)\right] = -2c_1 + c_2 \Rightarrow c_2 = 2c_1 = 4$

$\boxed{\text{2 pts.}}$

Therefore, $\boxed{\boxed{x = 2e^{-2t}\cos(t) + 4e^{-2t}\sin(t).}}$

Problem 7. (10 points)

a. (2 pts.) Find $\mathcal{L}\{7+\cos(2t)\}$

$$\mathcal{L}\{7+\cos(2t)\} = 7\mathcal{L}\{1\} + \mathcal{L}\{\cos(2t)\} = 7\cdot\frac{1}{s} + \frac{s}{s^2+2^2} = \frac{7}{s} + \frac{s}{s^2+4}$$

b. (8 pts.) Find $\mathcal{L}^{-1}\left\{\dfrac{4s+8}{(s-2)(s^2+4)}\right\}$.

Use a partial fraction decomposition: $\dfrac{4s+8}{(s-2)(s^2+4)} = \dfrac{A}{s-2} + \dfrac{Bs+C}{s^2+4}$

Multiplying both sides of the equation $\dfrac{4s+8}{(s-2)(s^2+4)} = \dfrac{A}{s-2} + \dfrac{Bs+C}{s^2+4}$ by the denominator $(s-2)(s^2+4)$, we obtain $4s+8 = A\left(s^2+4\right) + (Bs+C)(s-2) = As^2 + 4A + Bs^2 - 2Bs + Cs - 2C$ $= (A+B)s^2 + (-2B+C)s + 4A - 2C \Rightarrow$
$A+B=0,\ -2B+C=4,\ 4A-2C=8 \Rightarrow A=2,\ B=-2,\ C=0.$ $\boxed{\text{6 pts.}}$ Therefore,
$\mathcal{L}^{-1}\left\{\frac{4s+8}{(s-2)(s^2+4)}\right\} = \mathcal{L}^{-1}\left\{\frac{2}{s-2} + \frac{-2s}{s^2+4}\right\} = 2\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} - 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} = \boxed{\boxed{2e^{2t} - 2\cos(2t)}}$ using the Laplace Transform table entries $\mathcal{L}^{-1}\left\{\dfrac{1}{s-a}\right\} = e^{at}$ and $\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+k^2}\right\} = \cos(kt)$ $\boxed{\text{2 pts.}}$

Problem 8. (15 points) *Use the Laplace Transform* to solve the following initial value problem:

$$x'' - 4x' = 6e^t \quad \text{with } x(0) = 0 \text{ and } x'(0) = -2.$$

Solutions to this IVP not using the Laplace transform method will not receive any credit. Primes denote derivatives with respect to t: $x' = dx/dt$ and $x'' = d^2x/dt^2$.

$x'' - 4x' = 6e^t \Rightarrow \mathcal{L}\{x''-4x'\} = \mathcal{L}\left\{6e^t\right\} \Rightarrow \mathcal{L}\{x''\} - 4\mathcal{L}\{x'\} = 6\mathcal{L}\left\{e^t\right\}$ $\boxed{\text{2 pts.}}$

$\Rightarrow s^2\mathcal{L}\{x\} - sx(0) - x'(0) - 4\left[s\mathcal{L}\{x\} - x(0)\right] = 6\dfrac{1}{s-1}$ $\boxed{\text{4 pts.}}$

$\Rightarrow s^2\mathcal{L}\{x\} - s\cdot 0 - (-2) - 4\left[s\mathcal{L}\{x\} - 0\right] = \dfrac{6}{s-1} \Rightarrow \left(s^2-4s\right)\mathcal{L}\{x\} = -2 + \dfrac{6}{s-1} = \dfrac{-2s+8}{s-1} \Rightarrow$

$\mathcal{L}\{x\} = \dfrac{-2s+8}{(s-1)(s^2-4s)} = \dfrac{-2s+8}{s(s-1)(s-4)} = \dfrac{-2(s-4)}{s(s-1)(s-4)} = \dfrac{-2}{s(s-1)}$ $\boxed{\text{1 pt.}}$ $\Rightarrow x = \mathcal{L}^{-1}\left\{\dfrac{-2}{s(s-1)}\right\}$.

Use a partial fraction decomposition: $\dfrac{-2}{s(s-1)} = \dfrac{A}{s} + \dfrac{B}{s-1}$

Multiplying both sides of the equation $\dfrac{-2}{s(s-1)} = \dfrac{A}{s} + \dfrac{B}{s-1}$ by the denominator $s(s-1)$, we obtain $-2 = A(s-1) + Bs = (A+B)s - A \Rightarrow$
$A+B=0,\ -A=-2 \Rightarrow A=2,\ B=-2.$ Therefore,
$\mathcal{L}^{-1}\left\{\dfrac{-2}{s(s-1)}\right\} = \mathcal{L}^{-1}\left\{\dfrac{2}{s} + \dfrac{-2}{s-1}\right\} = 2\mathcal{L}^{-1}\left\{\dfrac{1}{s}\right\} - 2\mathcal{L}^{-1}\left\{\dfrac{1}{s-1}\right\}$ $\boxed{\text{6 pts.}}$

Therefore, $x = 2\mathcal{L}^{-1}\left\{\dfrac{1}{s}\right\} - 2\mathcal{L}^{-1}\left\{\dfrac{1}{s-1}\right\} \Rightarrow x = 2(1) - 2e^t \Rightarrow \boxed{\boxed{x = 2 - 2e^t}}$ using the Laplace Transform table entries $\mathcal{L}^{-1}\left\{\dfrac{1}{s}\right\} = 1$ and $\mathcal{L}^{-1}\left\{\dfrac{1}{s-a}\right\} = e^{at}$ $\boxed{\text{2 pts.}}$