92.236 Engineering Differential Equations Final Exam Solutions
Fall 2013

Problem 1. (10 pts.)

Solve the following initial value problem: $x\dfrac{dy}{dx} = y + x^2, \quad y(1) = 0.$

This is a linear d.e. because y and dy/dx appear just to the first power, multiplied by functions of x alone. $\boxed{\text{2 pts.}}$

First write the equation in standard form:

$$x\frac{dy}{dx} = y + x^2 \Rightarrow \frac{dy}{dx} - \left(\frac{1}{x}\right) y = x \; \boxed{\text{1 pts.}}$$

Next, find the integrating factor: $\rho(x) = e^{\int -1/x \; dx} = e^{-\ln(x)} = x^{-1}$. $\boxed{\text{3 pts.}}$

Multiply both sides of the standard form of the d.e. by the integrating factor:

$$x^{-1}\left[\frac{dy}{dx} - \left(\frac{1}{x}\right) y\right] = x^{-1}(x) \Rightarrow x^{-1}\frac{dy}{dx} - x^{-2}y = 1. \; \boxed{\text{1 pt.}}$$

Use the Product Rule backwards to rewrite the d.e. as $\dfrac{d}{dx}\left[x^{-1}y\right] = 1.$ $\boxed{\text{1 pt.}}$

Integrating both sides, we obtain $x^{-1}y = \displaystyle\int 1 \; dx = x + c.$ $\boxed{\text{1 pts.}}$

$y(1) = 0 \Rightarrow 1^{-1}(0) = 1 + c \Rightarrow c = -1$ $\boxed{\text{1 pt.}}$

Therefore, $x^{-1}y = x - 1$, so $\boxed{\boxed{y = x^2 - x.}}$

Problem 2. (10 pts.)

Solve the following initial value problem: $\dfrac{dy}{dx} = \dfrac{2xy}{y^2+1}$ with $y(0) = 1.$

This is a separable d.e. $\boxed{\text{2 pts.}}$ $\quad \dfrac{dy}{dx} = \dfrac{2xy}{y^2+1} \Rightarrow \dfrac{y^2+1}{y}\, dy = 2x \; dx$ $\boxed{\text{1 pt.}}$

$$\Rightarrow \int \frac{y^2+1}{y}\, dy = \int 2x \; dx \Rightarrow \int \left[\underbrace{\frac{y^2}{y}}_{=y} + \frac{1}{y}\right] dy = x^2 + c \Rightarrow \frac{y^2}{2} + \ln(y) = x^2 + c \; \boxed{\text{6 pts.}}$$

$y(0) = 1 \Rightarrow \dfrac{(1)^2}{2} + \ln(1) = 0^2 + c \Rightarrow c_1 = \dfrac{1}{2}$ $\boxed{\text{1 pt.}}$

Therefore, $\dfrac{y^2}{2} + \ln(y) = x^2 + \dfrac{1}{2} \Rightarrow \boxed{\boxed{y^2 + 2\ln(y) = 2x^2 + 1.}}$

Problem 3. (15 points)

A cup of coffee at temperature 125° F is brought into a room where the temperature is 75° F. After 5 minutes the coffee temperature is 120° F. When will the coffee temperature reach 100° F?

Recall that the de modeling heating/cooling problems is $\dfrac{dT}{dt} = -k\,(T - A).$

As we showed in class, the solution of the IVP $\dfrac{dT}{dt} = -k\,(T - A)\,,\; T(0) = T_0$ is
$T = A + (T_0 - A)\,e^{-kt}$ $\boxed{\text{3 pts.}}$

In this problem, $A = 75$ and $T_0 = 125$, so $T = 75 + (125 - 75)\,e^{-kt} = 75 + 50e^{-kt}$ $\boxed{\text{2 pts.}}$

$$T(5) = 120 \Rightarrow 120 = 75 + 50e^{-k(5)} \Rightarrow 45 = 50e^{-5k} \Rightarrow \underbrace{\frac{45}{50}}_{=0.9} = e^{-5k} \Rightarrow \ln(0.9) = \ln\left(e^{-5k}\right) = -5k \Rightarrow$$

$$k = -\frac{\ln(0.9)}{5} \quad \boxed{\text{5 pts.}}$$

Let t_f denote the time when the coffee temperature reaches 100° F.

$$T(t_f) = 100 \Rightarrow 100 = 75 + 50e^{-kt_f} \Rightarrow 25 = 50e^{-kt_f} \Rightarrow \underbrace{\frac{25}{50}}_{=0.5} = e^{-kt_f} \Rightarrow \ln(0.5) = \ln\left(e^{-kt_f}\right) = -kt_f \Rightarrow$$

$$\boxed{t_f = -\frac{\ln(0.5)}{k} = \frac{5\ln(0.5)}{\ln(0.9)} \approx 33 \text{ minutes}} \quad \boxed{\text{5 pts.}}$$

Problem 4. (10 pts.) Find the general solution to each of the following differential equations.

a. (4 points) $y'' - 5y' + 6y = 0$

The characteristic equation is $r^2 - 5r + 6 = 0 \Rightarrow (r-2)(r-3) = 0 \Rightarrow r = 2$ or $r = 3$ $\boxed{\text{2 pts.}}$

Therefore, $\boxed{y = c_1e^{2x} + c_2e^{3x}}$. $\boxed{\text{2 pts.}}$

b. (6 points) $y^{(4)} + 2y^{(3)} + 2y'' = 0$

The characteristic equation is $r^4 + 2r^3 + 2r = 0 \Rightarrow r^2\left(r^2 + 2r + 2\right) = 0.$

$$r^2 + 2r + 2 = 0 \Rightarrow r = \frac{-2 \pm \sqrt{2^2 - 4(1)(2)}}{2(1)} = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

Thus, the roots of the characteristic equation are 0 (double root) and $-1 \pm i$. $\boxed{\text{2 pts.}}$

Therefore, $y = c_1e^{0x} + c_2xe^{0x} + c_3e^{-1x}\cos(1x) + c_4e^{-1x}\sin(1x)$ or

$\boxed{y = c_1 + c_2x + c_3e^{-x}\cos(x) + c_4e^{-x}\sin(x)}$. $\boxed{\text{4 pts.}}$

Problem 5. (15 points)

Solve the following initial value problem:

$$y'' + y' - 6y = 50\cos(x) \quad \text{with } y(0) = -5 \text{ and } y'(0) = 5.$$

Note: $y' = dy/dx$ and $y'' = d^2y/dx^2$

Step 1. Find y_c by solving the homogeneous d.e. $y'' + y' - 6y = 0$.

Characteristic equation: $r^2 + r - 6 = 0 \Rightarrow (r+3)(r-2) = 0 \Rightarrow r = -3$ or $r = 2$.

Therefore, $y_c = c_1e^{-3x} + c_2e^{2x}$. $\boxed{\text{3 pts.}}$

Step 2. Find y_p using **either** the Method of Undetermined Coefficients **or** the Method of Variation of Parameters.

Method 1. Undetermined Coefficients.

Since the nonhomogeneous term in the d.e. $(50\cos(x))$ is a cosine function, we guess that y_p is the sum a cosine and sine with the same frequency as the cosine function in the nonhomogeneous term: $y_p = A\cos(x) + B\sin(x)$. No term in this guess duplicates a term in y_c, so there is no need to modify the guess. $\boxed{\text{3 pts.}}$

$y = A\cos(x) + B\sin(x) \Rightarrow y' = -A\sin(x) + B\cos(x) \Rightarrow y'' = -A\cos(x) - B\sin(x)$.

Therefore, the left side of the d.e. is
$y''+y'-6y = -A\cos(x)-B\sin(x)+[-A\sin(x)+B\cos(x)]-6\left[A\cos(x)+B\sin(x)\right] = [-7A+B]\cos(x)+[-A-7B]\sin(x)$. We want this to equal the nonhomogeneous term $50\cos(x)$, so $-7A+B = 50$, and $-A-7B=0 \Rightarrow$

$A=-7$, and $B=1$. Therefore, $y_p = -7\cos(x)+\sin(x)$. $\boxed{\text{6 pts.}}$

Method 2. Variation of Parameters.

From y_c we obtain two independent solutions of the homogeneous d.e: $y_1 = e^{-3x}$ and $y_2 = e^{2x}$. The Wronskian is given by $W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-3x} & e^{2x} \\ -3e^{-3x} & 2e^{2x} \end{vmatrix}$

$= e^{-3x}\left(2e^{2x}\right) - (-3e^{-3x})(e^{2x}) = 2e^{-x}+3e^{-x} = 5e^{-x}$. $\boxed{\text{1 pt.}}$

$$u_1 = \int \frac{-y_2 f(x)}{W(x)}\,dx = -\int \frac{e^{2x}\left(50\cos(x)\right)}{5e^{-x}}\,dx = -10\int e^{3x}\cos(x)\,dx$$

$= -10\left[\dfrac{e^{3x}}{3^2+1^2}\left(3\cos(x)+\sin(x)\right)\right] = e^{3x}\left[-3\cos(x)-\sin(x)\right]$ using entry 50 from the Table of Integrals. $\boxed{\text{3 pts.}}$

$$u_2 = \int \frac{y_1 f(x)}{W(x)}\,dx = \int \frac{e^{-3x}\left(50\cos(x)\right)}{5e^{-x}}\,dx = 10\int e^{-2x}\cos(x)\,dx$$

$= 10\left[\dfrac{e^{-2x}}{(-2)^2+1^2}\left(-2\cos(x)+\sin(x)\right)\right] = e^{-2x}\left[-4\cos(x)+2\sin(x)\right]$ again using entry 50 from the Table of Integrals. $\boxed{\text{3 pts.}}$

Therefore, $y_p = u_1y_1+u_2y_2 = \left\{e^{3x}\left[-3\cos(x)-\sin(x)\right]\right\}e^{-3x} + \left\{e^{-2x}\left[-4\cos(x)+2\sin(x)\right]\right\}e^{2x}$

$= [-3\cos(x)-\sin(x)] + [-4\cos(x)+2\sin(x)] = -7\cos(x)+\sin(x)$ $\boxed{\text{2 pts.}}$

Step 3. $y = y_c + y_p$, so $y = c_1e^{-3x}+c_2e^{2x}-7\cos(x)+\sin(x)$. $\boxed{\text{1 pt.}}$

Step 4. Use the initial conditions to find c_1 and c_2.

$y = c_1e^{-3x}+c_2e^{2x}-7\cos(x)+\sin(x) \Rightarrow y' = -3c_1e^{-3x}+2c_2e^{2x}+7\sin(x)+\cos(x)$

$y(0) = -5 \Rightarrow -5 = c_1e^0+c_2e^0-7\cos(0)+\sin(0) = c_1+c_2-7 \Rightarrow c_1+c_2 = 2$

$y'(0) = 5 \Rightarrow 5 = -3c_1e^0+2c_2e^0+7\sin(0)+\cos(0) = -3c_1+2c_2+1 \Rightarrow -3c_1+2c_2 = 4$

$c_1+c_2=2,\ -3c_1+2c_2=4 \Rightarrow c_1=0,\ c_2=2$ $\boxed{\text{2 pts.}}$

Therefore, $\boxed{\boxed{y = 2e^{2x}-7\cos(x)+\sin(x).}}$

Problem 6. (15 points)

Find the position function $x(t)$ for an unforced, damped mass-spring system with mass 1 kg, damping coefficient 6 Ns/m, and spring constant 9 N/m. Take $x(0) = 1$ m and $x'(0) = 0$ m/s.

The d.e. modeling an unforced mass-spring system is $mx''+cx'+kx=0$, where m denotes the mass of the object, c denotes the damping coefficient, and k denotes the spring constant. Substituting the given parameter values, we obtain the equation $x''+6x'+9x=0$ $\boxed{\text{5 pts.}}$

The characteristic equation of this d.e. is $r^2+6r+9=0 \Rightarrow (r+3)^2=0 \Rightarrow r=-3$ (double root). $\boxed{\text{4 pts.}}$

Therefore, $x = c_1e^{-3t}+c_2te^{-3t}$ $\boxed{\text{4 pts.}}$ so $x' = -3c_1e^{-3t}+c_2\left[e^{-3t}-3te^{-3t}\right]$

$x(0) = 1 \Rightarrow 1 = c_1e^0+c_2(0)e^0 = c_1 \Rightarrow c_1 = 1$

$x'(0) = 0 \Rightarrow 0 = -3c_1 e^0 + c_2\left[e^0 - 3(0)e^0\right] = -3c_1 + c_2 \Rightarrow c_2 = 3c_1 = 3$

Therefore, $\boxed{\boxed{x = e^{-3t} + 3te^{-3t}}}$ $\boxed{\text{2 pts.}}$

Problem 7. (10 points)

a. (2 pts.) Find $\mathcal{L}\left\{t^2 + \sin(3t)\right\}$

Using the Laplace Transform table entries for t^n and $\sin(kt)$, we find that

$$\mathcal{L}\left\{t^2 + \sin(3t)\right\} = \mathcal{L}\left\{t^2\right\} + \mathcal{L}\left\{\sin(3t)\right\} = \frac{2!}{s^{2+1}} + \frac{3}{s^2 + 3^2} \boxed{\boxed{= \frac{2}{s^3} + \frac{3}{s^2 + 9}}} \boxed{\text{2 pts.}}$$

b. (8 pts.) Find $\mathcal{L}^{-1}\left\{\dfrac{4s + 12}{(s+1)(s^2+1)}\right\}$.

Use a partial fraction decomposition: $\dfrac{4s + 12}{(s+1)(s^2+1)} = \dfrac{A}{s+1} + \dfrac{Bs + C}{s^2 + 1} \Rightarrow$

$$(s+1)\left(s^2+1\right)\left[\frac{4s+12}{(s+1)(s^2+1)}\right] = (s+1)\left(s^2+1\right)\left[\frac{A}{s+1} + \frac{Bs+C}{s^2+1}\right] \Rightarrow$$

$$4s + 12 = A(s^2+1) + (Bs + C)(s+1) = (A+B)s^2 + (B+C)s + (A+C)$$

$$\Rightarrow A + B = 0,\ B + C = 4,\ A + C = 12 \Rightarrow A = 4,\ B = -4,\ C = 8 \Rightarrow \boxed{\text{5 pts.}}$$

$$\mathcal{L}^{-1}\left\{\frac{4s+12}{(s+1)(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{4}{s+1} + \frac{-4s+8}{s^2+1}\right\} = 4\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + -4\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + 8\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} =$$

$\boxed{\boxed{4e^{-t} - 4\cos(t) + 8\sin(t)}}$ $\boxed{\text{3 pts.}}$

Problem 8. (15 points)

Use the Laplace Transform to solve the following initial value problem:

$$x'' - x = 6e^{2t} \quad \text{with } x(0) = 0 \text{ and } x'(0) = 2.$$

Solutions to this IVP not using the Laplace transform method will not receive any credit. Primes denote derivatives with respect to t: $x' = dx/dt$ and $x'' = d^2x/dt^2$.

$$x'' - x = 6e^{2t} \Rightarrow \mathcal{L}\left\{x'' - x\right\} = \mathcal{L}\left\{6e^{2t}\right\} \Rightarrow \mathcal{L}\left\{x''\right\} - \mathcal{L}\{x\} = 6\mathcal{L}\left\{e^{2t}\right\} \boxed{\text{3 pts.}}$$

$$\Rightarrow \left[s^2\mathcal{L}\{x\} - sx(0) - x'(0)\right] - 4\mathcal{L}\{x\} = 6\frac{1}{s-2} \boxed{\text{3 pts.}}$$

$$\left[s^2\mathcal{L}\{x\} - s\cdot 0 - 2\right] - \mathcal{L}\{x\} = \frac{6}{s-2} \Rightarrow \left(s^2 - 1\right)\mathcal{L}\{x\} = \frac{6}{s-2} + 2 = \frac{6 + 2(s-2)}{s-2} = \frac{2s+2}{s-2} \Rightarrow$$

$$\mathcal{L}\{x\} = \frac{2s+2}{(s^2-1)(s-2)} = \frac{2(s+1)}{[(s+1)(s-1)](s-2)} = \frac{2}{(s-1)(s-2)} \boxed{\text{1 pt.}} \Rightarrow x = \mathcal{L}^{-1}\left\{\frac{2}{(s-1)(s-2)}\right\}.$$

Use a partial fraction decomposition: $\dfrac{2}{(s-1)(s-2)} = \dfrac{A}{s-1} + \dfrac{B}{s-2}$

$$(s-1)(s-2)\left[\frac{2}{(s-1)(s-2)}\right] = (s-1)(s-2)\left[\frac{A}{s-1} + \frac{B}{s-2}\right] \Rightarrow$$

$$2 = A(s-2) + B(s-1) = (A+B)s + (-2A - B) \Rightarrow A + B = 0,\ -2A - B = 2 \Rightarrow A = -2,\ B = 2.$$

Therefore, $x = \mathcal{L}^{-1}\left\{\dfrac{-2}{s-1} + \dfrac{2}{s-2}\right\} = -2\mathcal{L}^{-1}\left\{\dfrac{1}{s-1}\right\} + 2\mathcal{L}^{-1}\left\{\dfrac{1}{s-2}\right\} \Rightarrow \boxed{\boxed{x = -2e^t + 2e^{2t}}}$ $\boxed{\text{8 pts.}}$