92.236 Engineering Differential Equations Exam # 1 Solutions Fall 2016

Problem 1. (10 points)

Is $y(x) = x^2$ is a solution of the d.e. $xyy' - x^4 = y^2$? Why or why not?

Left side of d.e: $y = x^2 \Rightarrow y' = 2x \Rightarrow xyy' - x^4 = x\left(x^2\right)(2x) - x^4 = 2x^4 - x^4 = x^4$. $\boxed{\text{4 pts.}}$

Right side of d.e: $y = x^2 \Rightarrow y^2 = \left(x^2\right)^2 = x^4$. $\boxed{\text{3 pts.}}$

Left side = right side, so $\boxed{\boxed{y(x) = x^2 \textbf{ is } \text{a solution of the d.e. } xyy' - x^4 = y^2}}$ $\boxed{\text{3 pts.}}$

Problem 2. (15 points)

A cup of coffee at a temperature of 150°F is brought into a room where the temperature is 75°F. After 10 minutes the temperature of the coffee is 140°F. What will the temperature of the coffee be after 30 minutes?

Let t denote time in minutes, let T denote the temperature of the can of soda in °F, and let A denote room temperature °F. As we showed in class, $T = A + (T_0 - A)\,e^{-kt}$ where $T_0 = T(0)$. Here $T_0 = 150°\text{F}$ and $A = 75°\text{F}$, so $T = 75 + (150 - 75)\,e^{-kt} = 75 + 75e^{-kt}$. $\boxed{\text{6 pts.}}$

$T(10) = 140 \Rightarrow 140 = 75 + 75e^{-k(10)} \Rightarrow 140 - 75 = 75e^{-10k} \Rightarrow \dfrac{65}{75} = e^{-10k} \Rightarrow$

$\ln(13/15) = \ln\left(e^{-10k}\right) = -10k \Rightarrow k = -\dfrac{\ln(13/15)}{10}$ $\boxed{\text{6 pts.}}$

Therefore, $T(30) = 75 + 75e^{-k(30)} = 75 + 75e^{30(\ln(13/15)/10)} \Rightarrow \boxed{\boxed{T(30) = 75 + 75e^{3\ln(13/15)} \approx 124°\text{F}}}$

$\boxed{\text{3 pts.}}$

Problem 3. (25 points)

Solve the following initial value problem:

$$x^2\frac{dy}{dx} = 12 - 5xy, \quad y(1) = 2.$$

This is a linear d.e. because y and dy/dx appear just to the first power, multiplied by functions of x alone. $\boxed{\text{5 pts.}}$

First write the equation in standard form:

$x^2\dfrac{dy}{dx} = 12 - 5xy \Rightarrow x^2\dfrac{dy}{dx} + 5xy = 12 \Rightarrow \dfrac{dy}{dx} + \left(\dfrac{5}{x}\right)y = \dfrac{12}{x^2}$ $\boxed{\text{3 pts.}}$

Next, find the integrating factor: $\rho(x) = e^{\int 5/x\ dx} = e^{5\ln(x)} = x^5$. $\boxed{\text{6 pts.}}$

Multiply both sides of the standard form of the d.e. by the integrating factor:

$x^5\left[\dfrac{dy}{dx} + \left(\dfrac{5}{x}\right)y\right] = x^5\left(\dfrac{12}{x^2}\right) \Rightarrow x^5\dfrac{dy}{dx} + 5x^4y = 12x^3$. $\boxed{\text{2 pts.}}$

Use the Product Rule backwards to rewrite the d.e. as $\dfrac{d}{dx}\left[x^5y\right] = 12x^3$. $\boxed{\text{4 pts.}}$

Integrating both sides, we obtain $x^5 y = \int 12x^3\ dx = 3x^4 + c$. $\boxed{\text{3 pts.}}$

$y(1) = 2 \Rightarrow 1^5(2) = 3(1)^4 + c \Rightarrow c = -1$ $\boxed{\text{2 pts.}}$

Therefore, $x^5 y = 3x^4 - 1$, so $\boxed{\boxed{y = \frac{3}{x} - \frac{1}{x^5}.}}$

Problem 4. (25 points)

Solve the following initial value problem:

$$3y^2 \frac{dy}{dx} = 1 + 2x, \quad y(2) = 1.$$

This is a separable d.e. $\boxed{\text{5 pts.}}$

$3y^2 \frac{dy}{dx} = 1 + 2x \Rightarrow 3y^2\ dy = (1 + 2x)\ dx$. $\boxed{\text{5 pts.}}$

$\Rightarrow \int 3y^2\ dy = \int (1 + 2x)\ dx \Rightarrow y^3 = x + x^2 + c$. $\boxed{\text{12 pts.}}$

$y(2) = 1 \Rightarrow 1^3 = 2 + 2^2 + c \Rightarrow c = -5$ $\boxed{\text{3 pts.}}$

$\Rightarrow y^3 = x + x^2 - 5 \Rightarrow \boxed{\boxed{y = \left(x + x^2 - 5\right)^{1/3}}}$.

Problem 5. (15 points)

A tank initially contains 50 liters of water in which 1 kilogram of salt is dissolved. A salt solution containing 0.1 kilogram of salt per liter is pumped into the tank at the rate of 10 liters per minute, and the well-mixed solution is pumped out of the tank at the rate of 6 liters per minute.

Let t denote time (in minutes), and let x denote the amount of salt in the tank at time t (in kilograms). Write down the differential equation $\left(\frac{dx}{dt} = \text{something}\right)$ and initial condition describing this mixing problem.

DO NOT SOLVE THE INITIAL VALUE PROBLEM. JUST WRITE DOWN THE D.E. AND INITIAL CONDITION.

$\frac{dx}{dt} =$ rate in - rate out $\boxed{\text{3 pts.}}$

$=$ (flow rate in)(concentration in) - (flow rate out)(concentration out), $\boxed{\text{3 pts.}}$ so

$$\frac{dx}{dt} = \underbrace{\left(10\frac{\text{L}}{\text{min}}\right)}_{\boxed{\text{1 pt.}}} \underbrace{\left(0.1\frac{\text{kg}}{\text{L}}\right)}_{\boxed{\text{1 pt.}}} - \underbrace{\left(6\frac{\text{L}}{\text{min}}\right)}_{\boxed{\text{1 pt.}}} \underbrace{\left(\frac{x \text{ gm}}{(50 + 4t)\text{ L}}\right)}_{\boxed{\text{5 pts.}}}.$$

(The volume in the tank at time t is initial volume + t (flow rate in - flow rate out) $= 50 + (10 - 6)t$ liters.)

Initially there is 1 kg. of sugar in the tank, so $x(0) = 1$ $\boxed{\text{1 pt.}}$.

Therefore, the initial value problem describing this mixing problem is $\boxed{\boxed{\frac{dx}{dt} = 1 - \frac{6x}{50 + 4t} \quad \text{with } x(0) = 1.}}$