92.236 Engineering Differential Equations   Exam # 2 Solutions   Fall 2016

Problem 1. (20 points) Consider the autonomous differential equation $\dfrac{dx}{dt} = x^2 - x - 2$.

a. Find all critical points (equilibrium solutions) of this d.e.

$x^2 - x - 2 = 0 \Rightarrow (x+1)(x-2) = 0 \Rightarrow$ the equilibrium solutions are $x = -1$ and $x = 2$

3 pts.

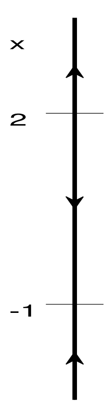
b. Draw the phase line (phase diagram) for this d.e. 8 pts.

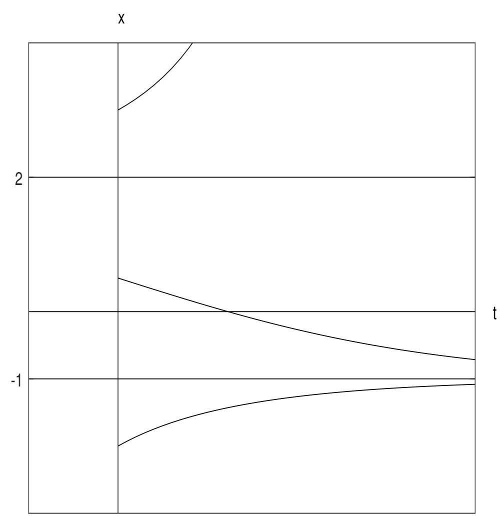
The two equilibrium solutions divide the phase line into 3 intervals: $x > 2$, $-1 < x < 2$, and $x < -1$.

$\left.\dfrac{dx}{dt}\right|_{x=3} = (3+1)(3-2) > 0$, so the direction arrow points up for $x > 2$.

$\left.\dfrac{dx}{dt}\right|_{x=0} = (0+1)(0-2) < 0$, so the direction arrow points down for $-1 < x < 2$.

$\left.\dfrac{dx}{dt}\right|_{x=-2} = (-2+1)(-2-2) > 0$, so the direction arrow points up for $x < -1$.





c. Determine whether each critical point is stable or unstable.

From the phase line we can see that 2 is unstable and -1 is stable. 2 pts.

d. If $x(0) = 1$, what value will $x(t)$ approach as t increases?

Since 1 lies in the interval $-1 < x < 2$, we can see from the phase line that $x(t) \to -1$ as t increases. 3 pts.

e. Sketch typical solution curves of the given d.e. Please draw these curves in a graph separate from your phase line. Be sure to include graphs of all equilibrium solutions, and be sure to label the axes. 4 pts. See the figure above.

Problem 2. (20 points) Solve the following differential equations:

a. $y'' + 2y' + y = 0$

Characteristic equation: $r^2 + 2r + 1 = 0 \Rightarrow (r+1)^2 = 0 \Rightarrow$
$r = -1$ (repeated root). $\boxed{4 \text{ pts.}}$ Therefore, $\boxed{\boxed{y = c_1 e^{-x} + c_2 x e^{-x}}}$ $\boxed{6 \text{ pts.}}$

b. $y'' + 5y' + 6y = 0$

Characteristic equation: $r^2 + 5r + 6 = 0 \Rightarrow (r+2)(r+3) = 0 \Rightarrow$
$r = -2$ or $r = -3$. $\boxed{4 \text{ pts.}}$ Therefore, $\boxed{\boxed{y = c_1 e^{-2x} + c_2 e^{-3x}}}$ $\boxed{6 \text{ pts.}}$

Problem 3. (20 points) Solve the following initial value problem.

$$2xy\frac{dy}{dx} - 3x^2 - 2y^2 = 0, \quad y(1) = 2$$

$2xy\dfrac{dy}{dx} - 3x^2 - 2y^2 = 0 \Rightarrow \dfrac{dy}{dx} = \dfrac{3x^2 + 2y^2}{2xy}$. dy/dx equals a rational function, and every term has the same degree (2). Therefore, this d.e. is homogeneous. $\boxed{4 \text{ pts.}}$

We introduce the new variable $v = y/x$. In the d.e. we replace $\dfrac{dy}{dx}$ by $v + x\dfrac{dv}{dx}$ and we replace y by xv:

$$\frac{dy}{dx} = \frac{3x^2 + 2y^2}{2xy} \Rightarrow \underbrace{v + x\frac{dv}{dx} = \frac{3x^2 + 2(xv)^2}{2x(xv)}}_{\boxed{4 \text{ pts.}}} = \frac{x^2\left(3 + 2v^2\right)}{2x^2 v} = \frac{3 + 2v^2}{2v} \Rightarrow \underbrace{x\frac{dv}{dx} = \frac{3 + 2v^2}{2v} - v = \frac{3}{2v}}_{\boxed{3 \text{ pts.}}}$$

$$\underbrace{\Rightarrow 2v\ dv = \frac{3}{x}\ dx}_{\boxed{2 \text{ pts.}}} \Rightarrow \int 2v\ dv = \int \frac{3}{x}\ dx \Rightarrow \underbrace{v^2 = 3\ln(x) + c}_{\boxed{3 \text{ pts.}}} \Rightarrow \underbrace{\left(\frac{y}{x}\right)^2 = 3\ln(x) + c}_{\boxed{2 \text{ pts.}}}$$

The initial condition $y(1) = 2 \Rightarrow (2/1)^2 = 3\ln(1) + c \Rightarrow c = 4$ $\boxed{2 \text{ pts.}}$.

Therefore, $\left(\dfrac{y}{x}\right)^2 = 3\ln(x) + 4 \Rightarrow \dfrac{y}{x} = \sqrt{3\ln(x) + 4} \Rightarrow \boxed{\boxed{y = x\sqrt{3\ln(x) + 4}}}$.

Problem 4. (20 points) Solve the following initial value problem.

$$4x + y^2 + \left[2xy + 3y^2\right]\frac{dy}{dx} = 0, \quad y(2) = 1$$

This d.e. is not separable, linear, or homogeneous. Test whether the d.e. is exact:

$$\underbrace{4x + y^2}_{M} + \underbrace{\left(2xy + 3y^2\right)}_{N}\frac{dy}{dx} = 0$$

$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left[4x + y^2\right] = 2y.$ $\boxed{1 \text{ pt.}}$ $\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left[2xy + 3y^2\right] = 2y.$ $\boxed{1 \text{ pt.}}$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the d.e. is exact. $\boxed{3 \text{ pts.}}$ Therefore, the solution of the d.e. is $f(x, y) = c$, where the function f satisfies the conditions $\frac{\partial f}{\partial x} = M = 4x + y^2$ and $\frac{\partial f}{\partial y} = N = 2xy + 3y^2$.

$\frac{\partial f}{\partial x} = 4x + y^2 \Rightarrow f = \int \left(4x + y^2\right) \partial x = 2x^2 + xy^2 + g(y)$ $\boxed{6 \text{ pts.}}$

$\Rightarrow \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[2x^2 + xy^2 + g(y)\right] = 2xy + g'(y)$

But $\frac{\partial f}{\partial y} = N = 2xy + 3y^2 \Rightarrow 2xy + g'(y) = 2xy + 3y^2 \Rightarrow g'(y) = 3y^2 \Rightarrow g(y) = y^3 \Rightarrow$

$f = 2x^2 + xy^2 + y^3$ $\boxed{6 \text{ pts.}}$

Therefore, the solution of the d.e. is $2x^2 + xy^2 + y^3 = c$ $\boxed{2 \text{ pts.}}$

$y(2) = 1 \Rightarrow 2(2)^2 + 2(1)^2 + 1^3 = c \Rightarrow c = 11.$ $\boxed{1 \text{ pt.}}$

Therefore, the solution of the initial value problem is $\boxed{\boxed{2x^2 + xy^2 + y^3 = 11}}$

Problem 5. (10 points) Let P denote the population of a colony of tribbles. Suppose that the birth rate β (number of births per week per tribble) is proportional to $\sqrt{P}$ and that the death rate δ (number of deaths per week per tribble) equals 0. Suppose the initial population is 4, and after two weeks the population is 16. What is the population after 3 weeks?

$\frac{dP}{dt} = \beta P - \delta P = (k\sqrt{P})P - (0)\,P = kP^{3/2}.$ $\boxed{3 \text{ pts.}}$

This is a separable d.e: $\frac{dP}{dt} = kP^{3/2} \Rightarrow \frac{dP}{P^{3/2}} = k\ dt$

$\Rightarrow \int P^{-3/2}\ dP = \int k\ dt \Rightarrow -2P^{-1/2} = kt + c \Rightarrow -\frac{2}{\sqrt{P}} = kt + c.$ $\boxed{4 \text{ pts.}}$

$P(0) = 4 \Rightarrow -\frac{2}{\sqrt{4}} = k(0) + c \Rightarrow c = -1$ $\boxed{1 \text{ pt.}}$

$\Rightarrow -\frac{2}{\sqrt{P}} = kt - 1 \Rightarrow \frac{2}{\sqrt{P}} = 1 - kt$

$P(2) = 16 \Rightarrow \frac{2}{\sqrt{16}} = 1 - k(2) \Rightarrow \frac{1}{2} = 1 - 2k \Rightarrow k = \frac{1}{4}.$ $\boxed{1 \text{ pt.}}$

Therefore, $\frac{2}{\sqrt{P}} = 1 - \frac{t}{4} \Rightarrow \frac{2}{\sqrt{P(3)}} = 1 - \frac{3}{4} = \frac{1}{4} \Rightarrow \sqrt{P(3)} = 8 \Rightarrow \boxed{\boxed{P(3) = 64 \text{ tribbles}}}.$ $\boxed{1 \text{ pt.}}$