92.236 Engineering Differential Equations Exam # 3 Solutions Fall 2016

Problem 1. (20 pts.) Solve the following differential equations.

a. (8 pts.) $y'' - 4y' + 13y = 0$

Characteristic equation: $r^2 - 4r + 13 = 0 \Rightarrow r = \dfrac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(13)}}{2(1)} = \dfrac{4 \pm \sqrt{-36}}{2} = \dfrac{4 \pm 6i}{2} = 2 \pm 3i$

$\boxed{\text{4 pts.}}$

Therefore, $\boxed{\boxed{y = c_1 e^{2x}\cos(3x) + c_2 e^{2x}\sin(3x)}}$ $\boxed{\text{4 pts.}}$

b. (12 pts.) $y^{(4)} + y'' = 0$

Characteristic equation: $r^4 + r^2 = 0 \Rightarrow r^2\left(r^2+1\right) = 0 \Rightarrow r = \pm i$ or $r = 0$ (double root). $\boxed{\text{4 pts.}}$
Therefore, $y = c_1 e^{0x} + c_2 x e^{0x} + c_3 e^{0x}\cos(1x) + c_4 e^{0x}\sin(1x)$, or

$\boxed{\boxed{y = c_1 + c_2 x + c_3\cos(x) + c_4\sin(x)}}$ $\boxed{\text{8 pts.}}$

Problem 2. (25 pts.) Solve the following initial value problem:

$$y'' - 4y = 16x^2, \; y(0) = 0, \; y'(0) = 4.$$

Step 1. Find y_c by solving the homogeneous d.e. $y'' - 4y = 0$.
Characteristic equation: $r^2 - 4 = 0 \Rightarrow (r+2)(r-2) = 0 \Rightarrow r = -2$ or $r = 2$.
Therefore, $y_c = c_1 e^{-2x} + c_2 e^{2x}$. $\boxed{\text{5 pts.}}$

Step 2. Find y_p.
Method 1: Undetermined Coefficients. Since the nonhomogeneous term $16x^2$ in the given d.e. is a polynomial of degree 2, we should guess that y_p is a polynomial of degree 2:
$y_p = Ax^2 + Bx + C$. $\boxed{\text{4 pts.}}$ No term in this guess duplicates a term in y_c, so there is no need to modify this guess. $\boxed{\text{2 pts.}}$
$y = Ax^2 + Bx + C \Rightarrow y' = 2Ax + B \Rightarrow y'' = 2A$. Therefore, the left side of the d.e. is
$y'' - 4y = 2A - 4\left[Ax^2 + Bx + C\right] = -4Ax^2 - 4Bx + (2A - 4C)$. We want this to equal the nonhomogeneous term $16x^2$:
$-4Ax^2 - 4Bx + (2A - 4C) = 16x^2 \Rightarrow -4A = 16,\ -4B = 0,\ 2A - 4C = 0 \Rightarrow A = -4,\ B = 0,\ C = -2$.
Thus, $y_p = -4x^2 - 2$. $\boxed{\text{9 pts.}}$

Method 2: Variation of Parameters. From y_c we obtain two independent solutions of the homogeneous d.e: $y_1 = e^{-2x}$ and $y_2 = e^{2x}$. $\boxed{\text{1 pt.}}$ The Wronskian is given by

$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-2x} & e^{2x} \\ -2e^{-2x} & 2e^{2x} \end{vmatrix} = e^{-2x}\left(2e^{2x}\right) - \left(-2e^{-2x}\right)e^{2x} = 4.$ $\boxed{\text{1 pt.}}$

$u_1 = \displaystyle\int \frac{-y_2 f(x)}{W(x)}\,dx = -\int \frac{e^{2x}\left(16x^2\right)}{4}\,dx = -4\int \left[x^2 e^{2x}\right]\,dx =$
$-\left(2x^2 - 2x + 1\right)e^{2x}$ after 2 integrations by parts. $\boxed{\text{4 pts.}}$

$u_2 = \displaystyle\int \frac{y_1 f(x)}{W(x)}\,dx = \int \frac{e^{-2x}\left(16x^2\right)}{4}\,dx = 4\int \left[x^2 e^{-2x}\right]\,dx = -\left(2x^2 + 2x + 1\right)e^{-2x}$ after 2 integrations by parts. $\boxed{\text{4 pts.}}$

Therefore, $y_p = u_1 y_1 + u_2 y_2 = \left[-\left(2x^2 - 2x + 1\right)e^{2x}\right]e^{-2x} + \left[-\left(2x^2 + 2x + 1\right)e^{-2x}\right]e^{2x} =$
$-\left(2x^2 - 2x + 1\right) - \left(2x^2 + 2x + 1\right) = -4x^2 - 2$ $\boxed{\text{5 pts.}}$

Step 3. $y = y_c + y_p$, so $y = c_1 e^{-2x} + c_2 e^{2x} - 4x^2 - 2$. $\boxed{\text{3 pts.}}$

Step 4. Use the initial conditions to determine the values of the arbitrary constants in the general solution.
$y = c_1 e^{-2x} + c_2 e^{2x} - 4x^2 - 2 \Rightarrow y' = -2c_1 e^{-2x} + 2c_2 e^{2x} - 8x$.
$y(0) = 0 \Rightarrow 0 = c_1 e^0 + c_2 e^0 - 4(0)^2 - 2 = c_1 + c_2 - 2 \Rightarrow c_1 + c_2 = 2$
$y'(0) = 4 \Rightarrow 4 = -2c_1 e^0 + 2c_2 e^0 - 8(0) = -2c_1 + 2c_2 . c_1 + c_2 = 2,\ -2c_1 + 2c_2 = 4 \Rightarrow c_1 = 0,\ c_2 = 2$
$\boxed{\text{2 pts.}}$

Therefore, $\boxed{\boxed{y = 2e^{2x} - 4x^2 - 2}}$

Problem 3. (20 points) Consider a forced, damped mass-spring system with mass $m = 1$ kg, damping constant $c = 5$ N·s/m, spring constant $k = 4$ N/m, and external force $F_{\text{ext}} = 40\cos(2t)$ N. Find the steady-state (steady periodic) solution x_{sp}.

The d.e. describing a mass-spring system is $mx'' + cx' + kx = F_{\text{e}}(t)$. $\boxed{\text{2 pts.}}$
In this problem, the d.e. becomes $x'' + 5x' + 4x = 40\cos(2t)$. $\boxed{\text{2 pts.}}$

The steady periodic solution is the particular solution x_p. $\boxed{\text{4 pts.}}$ Since the nonhomogeneous term $40\cos(2t)$ is a cosine, we should guess that x_p is a combination of a cosine and a sine with the same frequency: $x_p = A\cos(2t) + B\sin(2t)$. $\boxed{\text{5 pts.}}$ (No part of this guess will duplicate part of x_c because x_c is a transient term containing decaying exponential functions.)
$x = A\cos(2t) + B\sin(2t) \Rightarrow x' = -2A\sin(2t) + 2B\cos(2t) \Rightarrow x'' = -4A\cos(2t) - 4B\sin(2t)$. Therefore, the left side of the d.e. is
$x'' + 5x' + 4x = -4A\cos(2t) - 4B\sin(2t) + 5[-2A\sin(2t) + 2B\cos(2t))] + 4[A\cos(2t) + B\sin(2t)]$
$= 10B\cos(2t) - 10A\sin(2t)$.
We want this to equal the nonhomogeneous term $40\cos(2t)$:
$10B\cos(2t) - 10A\sin(2t) = 40\cos(2t) \Rightarrow 10B = 40,\ -10A = 0 \Rightarrow A = 0$ and $B = 4$. Therefore, $\boxed{\boxed{x_{\text{sp}} = 4\sin(2t)}}$. $\boxed{\text{7 pts.}}$

Problem 4. (20 points) Solve the system $\begin{cases} x' &= x + 6y \\ y' &= x \end{cases}$

Note: $x' = dx/dt$ and $y' = dy/dt$. t is the independent variable.

Take the derivative of both sides of the second d.e. in the system: $y' = x \Rightarrow y'' = x'$. The first d.e. in the system is $x' = x + 6y$. Therefore, $y'' = x + 6y$. From the second d.e. in the system, $x = y'$, so we have $y'' = y' + 6y$ $\boxed{\text{8 pts.}}$

$y'' = y' + 6y \Rightarrow y'' - y' - 6y = 0$.
Characteristic equation: $r^2 - r - 6 = 0 \Rightarrow (r+2)(r-3) = 0 \Rightarrow r = -2$ or $r = 3 \Rightarrow y = c_1 e^{-2t} + c_2 e^{3t}$.
$\boxed{\text{8 pts.}}$

The second d.e. in the given system says $x = y'$, so $x = -2c_1 e^{-2t} + 3c_2 e^{3t}$. Therefore, the solution of the given system is $\boxed{\boxed{x = -2c_1 e^{-2t} + 3c_2 e^{3t},\ y = c_1 e^{-2t} + c_2 e^{3t}}}$ $\boxed{\text{4 pts.}}$