92.236 Engineering Differential Equations   Exam # 1 Solutions   Fall 2016

Problem 1. (10 points)

Is $y(x) = x^3$ is a solution of the d.e. $yy' - 2x^5 = x^2y$? Why or why not?

Left side of d.e: $y = x^3 \Rightarrow y' = 3x^2 \Rightarrow yy' - 2x^5 = x^3\left(3x^2\right) - 2x^5 = 3x^5 - 2x^5 = x^5$. $\boxed{\text{4 pts.}}$

Right side of d.e: $y = x^3 \Rightarrow x^2y = x^2\left(x^3\right)^2 = x^5$. $\boxed{\text{3 pts.}}$

Left side = right side, so $\boxed{\boxed{y(x) = x^3 \textbf{ is} \text{ a solution of the d.e. } yy' - 2x^5 = x^2y}}$ $\boxed{\text{3 pts.}}$

Problem 2. (15 points)

A cup of coffee at a temperature of 160°F is brought into a room where the temperature is 80°F. After 5 minutes the temperature of the coffee is 150°F. What will the temperature of the coffee be after 20 minutes?

Let t denote time in minutes, let T denote the temperature of the can of soda in °F, and let A denote room temperature °F. As we showed in class, $T = A + (T_0 - A)\, e^{-kt}$ where $T_0 = T(0)$. Here $T_0 = 160$°F and $A = 80$°F, so $T = 80 + (160 - 80)\, e^{-kt} = 80 + 80e^{-kt}$. $\boxed{\text{6 pts.}}$

$T(5) = 150 \Rightarrow 150 = 80 + 80e^{-k(5)} \Rightarrow 150 - 80 = 80e^{-5k} \Rightarrow \dfrac{70}{80} = e^{-5k} \Rightarrow$

$\ln(7/8) = \ln\left(e^{-5k}\right) = -5k \Rightarrow k = -\dfrac{\ln(7/8)}{5}$ $\boxed{\text{6 pts.}}$

Therefore, $T(20) = 80 + 80e^{-k(20)} = 80 + 80e^{20(\ln(7/8)/5)} \Rightarrow \boxed{\boxed{T(20) = 80 + 80e^{4\ln(7/8)} \approx 127°\text{F}}}$ $\boxed{\text{3 pts.}}$

Problem 3. (25 points)

Solve the following initial value problem:

$$x\frac{dy}{dx} = 6x^2 - 4y, \quad y(1) = 2.$$

This is a linear d.e. because y and dy/dx appear just to the first power, multiplied by functions of x alone. $\boxed{\text{5 pts.}}$

First write the equation in standard form:

$x\dfrac{dy}{dx} = 6x^2 - 4y \Rightarrow x\dfrac{dy}{dx} + 4y = 6x^2 \Rightarrow \dfrac{dy}{dx} + \left(\dfrac{4}{x}\right)y = 6x$ $\boxed{\text{3 pts.}}$

Next, find the integrating factor: $\rho(x) = e^{\int 4/x\; dx} = e^{4\ln(x)} = x^4$. $\boxed{\text{6 pts.}}$

Multiply both sides of the standard form of the d.e. by the integrating factor:

$x^4\left[\dfrac{dy}{dx} + \left(\dfrac{4}{x}\right)y\right] = x^4(6x) \Rightarrow x^4\dfrac{dy}{dx} + 4x^3y = 6x^5$. $\boxed{\text{2 pts.}}$

Use the Product Rule backwards to rewrite the d.e. as $\dfrac{d}{dx}\left[x^4y\right] = 6x^5$. $\boxed{\text{4 pts.}}$

Integrating both sides, we obtain $x^4y = \displaystyle\int 6x^5\; dx = x^6 + c$. $\boxed{\text{3 pts.}}$

$y(1) = 2 \Rightarrow 1^4(2) = (1)^6 + c \Rightarrow c = 1$ $\boxed{\text{2 pts.}}$

Therefore, $x^4 y = x^6 + 1$, so $\boxed{\boxed{y = x^2 + \frac{1}{x^4}.}}$

Problem 4. (25 points)

Solve the following initial value problem:

$$4y^3 \frac{dy}{dx} = 2 + 3x^2, \quad y(2) = 1.$$

This is a separable d.e. $\boxed{\text{5 pts.}}$

$4y^3 \frac{dy}{dx} = 2 + 3x^2 \Rightarrow 4y^3 \, dy = \left(2 + 3x^2\right) \, dx.$ $\boxed{\text{5 pts.}}$

$\Rightarrow \int 4y^3 \, dy = \int \left(2 + 3x^2\right) \, dx \Rightarrow y^4 = 2x + x^3 + c.$ $\boxed{\text{12 pts.}}$

$y(2) = 1 \Rightarrow 1^4 = 2(2) + 2^3 + c \Rightarrow c = -11$ $\boxed{\text{3 pts.}}$

$\Rightarrow y^4 = 2x + x^3 - 11 \Rightarrow \boxed{\boxed{y = \left(2x + x^3 - 11\right)^{1/4}}}.$

Problem 5. (15 points)

A tank initially contains 200 liters of water in which 2 kilograms of salt are dissolved. A salt solution containing 0.2 kilogram of salt per liter is pumped into the tank at the rate of 10 liters per minute, and the well-mixed solution is pumped out of the tank at the rate of 4 liters per minute.

Let t denote time (in minutes), and let x denote the amount of salt in the tank at time t (in kilograms). Write down the differential equation $\left(\frac{dx}{dt} = \text{something}\right)$ and initial condition describing this mixing problem.

DO NOT SOLVE THE INITIAL VALUE PROBLEM. JUST WRITE DOWN THE D.E. AND INITIAL CONDITION.

$\frac{dx}{dt} = \text{rate in - rate out}$ $\boxed{\text{3 pts.}}$

$=$ (flow rate in)(concentration in) - (flow rate out)(concentration out), $\boxed{\text{3 pts.}}$ so

$$\frac{dx}{dt} = \underbrace{\left(10 \frac{\text{L}}{\text{min}}\right)}_{\boxed{\text{1 pt.}}} \underbrace{\left(0.2 \frac{\text{kg}}{\text{L}}\right)}_{\boxed{\text{1 pt.}}} - \underbrace{\left(4 \frac{\text{L}}{\text{min}}\right)}_{\boxed{\text{1 pt.}}} \underbrace{\left(\frac{x \text{ gm}}{(200 + 6t) \text{ L}}\right)}_{\boxed{\text{5 pts.}}}.$$

(The volume in the tank at time t is initial volume + t (flow rate in - flow rate out) $= 200 + (10 - 4)t$ liters.)

Initially there are 2 kg. of salt in the tank, so $x(0) = 2$ $\boxed{\text{1 pt.}}$.

Therefore, the initial value problem describing this mixing problem is $\boxed{\boxed{\frac{dx}{dt} = 2 - \frac{4x}{200 + 6t} \text{ with } x(0) = 2.}}$