92.236 Engineering Differential Equations Exam # 2 Solutions Fall 2016

Problem 1. (20 points) Consider the autonomous differential equation $\dfrac{dx}{dt} = x^2 - x - 6$.

a. Find all critical points (equilibrium solutions) of this d.e.

$x^2 - x - 6 = 0 \Rightarrow (x+2)(x-3) = 0 \Rightarrow$ $\boxed{\boxed{\text{the equilibrium solutions are } x = -2 \text{ and } x = 3}}$

$\boxed{\text{3 pts.}}$

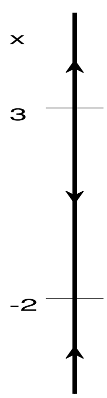
b. Draw the phase line (phase diagram) for this d.e. $\boxed{\text{8 pts.}}$

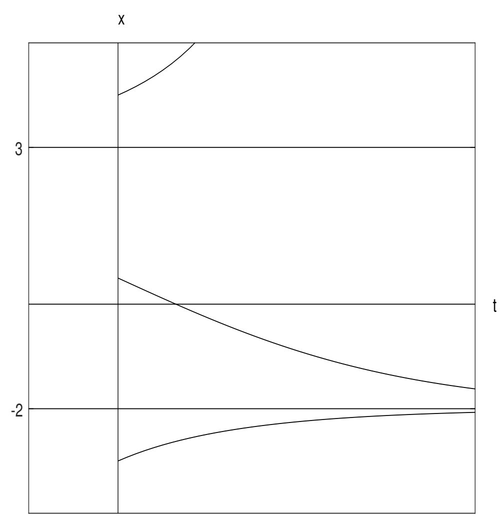
The two equilibrium solutions divide the phase line into 3 intervals: $x > 3$, $-2 < x < 3$, and $x < -2$.

$\left.\dfrac{dx}{dt}\right|_{x=4} = (4+2)(4-3)) > 0$, so the direction arrow points up for $x > 3$.

$\left.\dfrac{dx}{dt}\right|_{x=0} = (0+2)(0-3) < 0$, so the direction arrow points down for $-2 < x < 3$.

$\left.\dfrac{dx}{dt}\right|_{x=-3} = (-3+2)(-3-3) > 0$, so the direction arrow points up for $x < -2$.





c. Determine whether each critical point is stable or unstable.

From the phase line we can see that $\boxed{\boxed{-2 \text{ is stable and } 3 \text{ is unstable}}}$. $\boxed{\text{2 pts.}}$

d. If $x(0) = 1$, what value will $x(t)$ approach as t increases?

Since 1 lies in the interval $-2 < x < 3$, we can see from the phase line that $\boxed{\boxed{x(t) \to -2}}$ as t increases. $\boxed{\text{3 pts.}}$

e. Sketch typical solution curves of the given d.e. Please draw these curves in a graph separate from your phase line. Be sure to include graphs of all equilibrium solutions, and be sure to label the axes. $\boxed{\text{4 pts.}}$ See the figure above.

Problem 2. (20 points) Solve the following differential equations:

a. $y'' - 4y' + 3y = 0$

Characteristic equation: $r^2 - 4r + 3 = 0 \Rightarrow (r-1)(r-3) = 0 \Rightarrow$
$r = 1$ or $r = 3$. $\boxed{\text{4 pts.}}$ Therefore, $\boxed{\boxed{y = c_1 e^x + c_2 e^{3x}}}$ $\boxed{\text{6 pts.}}$

b. $y'' - 2y' + y = 0$

Characteristic equation: $r^2 - 2r + 1 = 0 \Rightarrow (r-1)^2 = 0 \Rightarrow$
$r = 1$ (repeated root). $\boxed{\text{4 pts.}}$ Therefore, $\boxed{\boxed{y = c_1 e^x + c_2 x e^x}}$ $\boxed{\text{6 pts.}}$

Problem 3. (20 points) Solve the following initial value problem.

$$8x^3 + y^3 + \left[3xy^2 + 2y\right] \frac{dy}{dx} = 0, \quad y(1) = 2$$

This d.e. is not separable, linear, or homogeneous. Test whether the d.e. is exact:

$$\underbrace{8x^3 + y^3}_{M} + \underbrace{\left(3xy^2 + 2y\right)}_{N} \frac{dy}{dx} = 0$$

$\dfrac{\partial M}{\partial y} = \dfrac{\partial}{\partial y}\left[8x^3 + y^3\right] = 3y^2.$ $\boxed{\text{1 pt.}}$ $\dfrac{\partial N}{\partial x} = \dfrac{\partial}{\partial x}\left[3xy^2 + 2y\right] = 3y^2.$ $\boxed{\text{1 pt.}}$
Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the d.e. is exact. $\boxed{\text{3 pts.}}$ Therefore, the solution of the d.e. is $f(x, y) = c$, where the function f satisfies the conditions $\dfrac{\partial f}{\partial x} = M = 8x^3 + y^3$ and $\dfrac{\partial f}{\partial y} = N = 3xy^2 + 2y$.
$\dfrac{\partial f}{\partial x} = 8x^3 + y^3 \Rightarrow f = \displaystyle\int \left(8x^3 + y^3\right)\, \partial x = 2x^4 + xy^3 + g(y)$ $\boxed{\text{6 pts.}}$
$\Rightarrow \dfrac{\partial f}{\partial y} = \dfrac{\partial}{\partial y}\left[2x^4 + xy^3 + g(y)\right] = 3xy^2 + g'(y)$
But $\dfrac{\partial f}{\partial y} = N = 3xy^2 + 2y \Rightarrow 3xy^2 + g'(y) = 3xy^2 + 2y \Rightarrow g'(y) = 2y \Rightarrow g(y) = y^2 \Rightarrow$
$f = 2x^4 + xy^3 + y^2$ $\boxed{\text{6 pts.}}$

Therefore, the solution of the d.e. is $2x^4 + xy^3 + y^2 = c$ $\boxed{\text{2 pts.}}$
$y(1) = 2 \Rightarrow 2(1)^4 + 1(2)^3 + 2^2 = c \Rightarrow c = 14.$ $\boxed{\text{1 pt.}}$

Therefore, the solution of the initial value problem is $\boxed{\boxed{2x^4 + xy^3 + y^2 = 14}}$

Problem 4. (20 points) Solve the following initial value problem.

$$2xy^2\frac{dy}{dx} + 3x^3 - 2y^3 = 0, \quad y(1) = 3$$

$2xy^2\dfrac{dy}{dx} + 3x^3 - 2y^3 = 0 \Rightarrow \dfrac{dy}{dx} = \dfrac{2y^3 - 3x^3}{2xy^2}$. dy/dx equals a rational function, and every term has the same degree (3). Therefore, this d.e. is homogeneous. $\boxed{\text{4 pts.}}$

We introduce the new variable $v = y/x$. In the d.e. we replace $\dfrac{dy}{dx}$ by $v + x\dfrac{dv}{dx}$ and we replace y by xv:

$$\frac{dy}{dx} = \frac{2y^3 - 3x^3}{2xy^2} \Rightarrow \underbrace{v + x\frac{dv}{dx} = \frac{2(xv)^3 - 3x^3}{2x(xv)^2}}_{\boxed{\text{4 pts.}}} = \frac{x^3(2v^3 - 3)}{2x^3v^2} = \frac{2v^3 - 3}{2v^2} \Rightarrow \underbrace{x\frac{dv}{dx} = \frac{2v^3 - 3}{2v^2} - v = -\frac{3}{2v^2}}_{\boxed{\text{3 pts.}}}$$

$$\underbrace{\Rightarrow v^2\,dv = -\frac{3}{2x}\,dx}_{\boxed{\text{2 pts.}}} \Rightarrow \int v^2\,dv = -\int \frac{3}{2x}\,dx \Rightarrow \underbrace{\frac{v^3}{3} = -\frac{3}{2}\ln(x) + c}_{\boxed{\text{3 pts.}}} \Rightarrow \underbrace{\frac{1}{3}\left(\frac{y}{x}\right)^3 = -\frac{3}{2}\ln(x) + c}_{\boxed{\text{2 pts.}}}$$

The initial condition $y(1) = 3 \Rightarrow \dfrac{1}{3}\left(\dfrac{3}{1}\right)^3 = -\dfrac{3}{2}\ln(1) + c \Rightarrow c = 9$ $\boxed{\text{2 pts.}}$.

Therefore, $\dfrac{1}{3}\left(\dfrac{y}{x}\right)^3 = -\dfrac{3}{2}\ln(x) + 9 \Rightarrow \left(\dfrac{y}{x}\right)^3 = -\dfrac{9}{2}\ln(x) + 27 \Rightarrow \dfrac{y}{x} = \left(-\dfrac{9}{2}\ln(x) + 27\right)^{1/3} \Rightarrow$

$$\boxed{\boxed{y = x\left(27 - \frac{9}{2}\ln(x)\right)^{1/3}}}.$$

Problem 5. (10 points) Let P denote the population of a colony of tribbles. Suppose that the birth rate β (number of births per week per tribble) is proportional to $\dfrac{1}{\sqrt{P}}$ and that the death rate δ (number of deaths per week per tribble) equals 0. Suppose the initial population is 4, and after two weeks the population is 16. What is the population after 3 weeks?

$\dfrac{dP}{dt} = \beta P - \delta P = \left(\dfrac{k}{\sqrt{P}}\right)P - (0)\,P = kP^{1/2}$. $\boxed{\text{3 pts.}}$

This is a separable d.e: $\dfrac{dP}{dt} = kP^{1/2} \Rightarrow \dfrac{dP}{P^{1/2}} = k\,dt$

$\Rightarrow \displaystyle\int P^{-1/2}\,dP = \int k\,dt \Rightarrow 2P^{1/2} = kt + c \Rightarrow 2\sqrt{P} = kt + c.$ $\boxed{\text{4 pts.}}$

$P(0) = 4 \Rightarrow 2\sqrt{4} = k(0) + c \Rightarrow c = 4$ $\boxed{\text{1 pt.}}$ $\Rightarrow 2\sqrt{P} = kt + 4$

$P(2) = 16 \Rightarrow 2\sqrt{16} = k(2) + 4 \Rightarrow 8 = 2k + 4 \Rightarrow k = 2.$ $\boxed{\text{1 pt.}}$

Therefore, $2\sqrt{P} = 2t + 4 \Rightarrow 2\sqrt{P(3)} = 2(3) + 4 = 10 \Rightarrow \sqrt{P(3)} = 5 \Rightarrow \boxed{\boxed{P(3) = 25 \text{ tribbles}}}$.

$\boxed{\text{1 pt.}}$