92.236 Engineering Differential Equations Exam # 3 Solutions Fall 2016

Problem 1. (20 pts.) Solve the following differential equations.

a. (8 pts.) $y'' - 6y' + 13y = 0$

Characteristic equation: $r^2 - 6r + 13 = 0 \Rightarrow r = \dfrac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(13)}}{2(1)} = \dfrac{6 \pm \sqrt{-16}}{2} = \dfrac{6 \pm 4i}{2} = 3 \pm 2i$

$\boxed{4 \text{ pts.}}$

Therefore, $\boxed{\boxed{y = c_1 e^{3x} \cos(2x) + c_2 e^{3x} \sin(2x)}}$ $\boxed{4 \text{ pts.}}$

b. (12 pts.) $y^{(4)} - y'' = 0.$

Characteristic equation: $r^4 - r^2 = 0 \Rightarrow r^2\left(r^2 - 1\right) = 0 \Rightarrow r^2(r+1)(r-1) = 0$ $r = 0$ (double root) or $r = -1$ or $r = 1$. $\boxed{4 \text{ pts.}}$ Therefore, $y = c_1 e^{0x} + c_2 x e^{0x} + c_3 e^{1x} + c_4 e^{-1x}$, or

$\boxed{\boxed{y = c_1 + c_2 x + c_3 e^{-x} + c_4 e^{x}}}$ $\boxed{8 \text{ pts.}}$

Problem 2. (25 pts.) Solve the following initial value problem:

$$y'' + 4y = 8x^2, \; y(0) = 1, \; y'(0) = 0.$$

Step 1. Find y_c by solving the homogeneous d.e. $y'' + 4y = 0$.
Characteristic equation: $r^2 + 4 = 0 \Rightarrow r^2 = -4 \Rightarrow r = \pm\sqrt{-4} = 0 \pm 2i$.
Therefore, $y_c = c_1 e^{0x} \cos(2x) + c_2 e^{0x} \sin(2x) = c_1 \cos(2x) + c_2 \sin(2x)$. $\boxed{5 \text{ pts.}}$

Step 2. Find y_p.
Method 1: Undetermined Coefficients. Since the nonhomogeneous term $8x^2$ in the given d.e. is a polynomial of degree 2, we should guess that y_p is a polynomial of degree 2:
$y_p = Ax^2 + Bx + C$. $\boxed{4 \text{ pts.}}$ No term in this guess duplicates a term in y_c, so there is no need to modify this guess. $\boxed{2 \text{ pts.}}$
$y = Ax^2 + Bx + C \Rightarrow y' = 2Ax + B \Rightarrow y'' = 2A$. Therefore, the left side of the d.e. is
$y'' + 4y = 2A + 4\left[Ax^2 + Bx + C\right] = 4Ax^2 + 4Bx + (2A + 4C)$. We want this to equal the nonhomogeneous term $8x^2$:
$4Ax^2 + 4Bx + (2A + 4C) = 8x^2 \Rightarrow 4A = 8, \; 4B = 0, \; 2A + 4C = 0 \Rightarrow A = 2, \; B = 0, \; C = -1$.
Thus, $y_p = 2x^2 - 1$. $\boxed{9 \text{ pts.}}$

Method 2: Variation of Parameters. From y_c we obtain two independent solutions of the homogeneous d.e: $y_1 = \cos(2x)$ and $y_2 = \sin(2x)$. $\boxed{1 \text{ pt.}}$ The Wronskian is given by

$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos(2x) \end{vmatrix} = \cos(2x)\left(2\cos(2x)\right) - \left(-2\sin(2x)\right)\sin(2x) =$
$2\left[\cos^2(2x) + \sin^2(2x)\right] = 2.$ $\boxed{1 \text{ pt.}}$

$u_1 = \displaystyle\int \frac{-y_2 f(x)}{W(x)}\, dx = -\int \frac{\sin(2x)\left(8x^2\right)}{2}\, dx = -4\int \left[x^2 \sin(2x)\right]\, dx =$
$\left(2x^2 - 1\right)\cos(2x) - 2x\sin(2x)$ after 2 integrations by parts. $\boxed{4 \text{ pts.}}$

$u_2 = \displaystyle\int \frac{y_1 f(x)}{W(x)}\, dx = \int \frac{\cos(2x)\left(8x^2\right)}{2}\, dx = 4\int \left[x^2 \cos(2x)\right]\, dx = \left(2x^2 - 1\right)\sin(2x) + 2x\sin(2x)$ after 2 integrations by parts. $\boxed{4 \text{ pts.}}$

Therefore, $y_p = u_1 y_1 + u_2 y_2 =$
$\left[\left(2x^2-1\right)\cos(2x) - 2x\sin(2x)\right]\cos(2x) + \left[\left(2x^2-1\right)\sin(2x) + 2x\sin(2x)\right]\sin(2x) =$
$\left(2x^2-1\right)\cos^2(2x) - 2x\sin(2x)\cos(2x) + \left(2x^2-1\right)\sin^2(2x) + 2x\sin(2x)\cos(2x) =$
$\left(2x^2-1\right)\left[\cos^2(2x)+\sin^2(2x)\right] = 2x^2 - 1$ $\boxed{\text{5 pts.}}$

Step 3. $y = y_c + y_p$, so $y = c_1\cos(2x) + c_2\sin(2x) + 2x^2 - 1.$ $\boxed{\text{3 pts.}}$

Step 4. Use the initial conditions to determine the values of the arbitrary constants in the general solution.
$y = c_1\cos(2x) + c_2\sin(2x) + 2x^2 - 1 \Rightarrow y' = -2c_1\sin(2x) + 2c_2\cos(2x) + 4x.$
$y(0) = 1 \Rightarrow 1 = c_1\cos(0) + c_2\sin(0) + 2(0)^2 - 1 = c_1 - 1 \Rightarrow c_1 = 2$
$y'(0) = 0 \Rightarrow 0 = -2c_1\sin(0) + 2c_2\cos(0) + 4(0) = 2c_2 \Rightarrow c_2 = 0$
$\boxed{\text{2 pts.}}$

Therefore, $\boxed{\boxed{y = 2\cos(2x) + 2x^2 - 1}}$

Problem 3. (20 points) Consider a forced, damped mass-spring system with mass $m = 1$ kg, damping constant $c = 10$ N·s/m, spring constant $k = 9$ N/m, and external force $F_{\text{ext}} = 60\cos(3t)$ N. Find the steady-state (steady periodic) solution x_{sp}.

The d.e. describing a mass-spring system is $mx'' + cx' + kx = F_{\text{e}}(t)$. $\boxed{\text{2 pts.}}$
In this problem, the d.e. becomes $x'' + 10x' + 9x = 60\cos(3t)$. $\boxed{\text{2 pts.}}$

The steady periodic solution is the particular solution x_p. $\boxed{\text{4 pts.}}$ Since the nonhomogeneous term $60\cos(3t)$ is a cosine, we should guess that x_p is a combination of a cosine and a sine with the same frequency: $x_p = A\cos(3t) + B\sin(3t)$. $\boxed{\text{5 pts.}}$ (No part of this guess will duplicate part of x_c because x_c is a transient term containing decaying exponential functions.)
$x = A\cos(3t) + B\sin(3t) \Rightarrow x' = -3A\sin(3t) + 3B\cos(3t) \Rightarrow x'' = -9A\cos(3t) - 9B\sin(3t)$. Therefore, the left side of the d.e. is
$x'' + 10x' + 9x = -9A\cos(3t) - 9B\sin(3t) + 10\left[-3A\sin(3t) + 3B\cos(3t))\right] + 9\left[A\cos(3t) + B\sin(3t)\right]$
$= 30B\cos(2t) - 30A\sin(2t)$.
We want this to equal the nonhomogeneous term $60\cos(3t)$:
$30B\cos(2t) - 30A\sin(2t) = 60\cos(3t) \Rightarrow 30B = 60,\ -30A = 0 \Rightarrow A = 0$ and $B = 2$. Therefore,
$\boxed{\boxed{x_{\text{sp}} = 2\sin(3t)}}$. $\boxed{\text{7 pts.}}$

Problem 4. (20 points) Solve the system $\begin{cases} x' &= x + 12y \\ y' &= x \end{cases}$

Note: $x' = dx/dt$ and $y' = dy/dt$. t is the independent variable.

Take the derivative of both sides of the second d.e. in the system: $y' = x \Rightarrow y'' = x'$. The first d.e. in the system is $x' = x + 12y$. Therefore, $y'' = x + 12y$. From the second d.e. in the system, $x = y'$, so we have $y'' = y' + 12y$ $\boxed{\text{8 pts.}}$

$y'' = y' + 12y \Rightarrow y'' - y' - 12y = 0.$
Characteristic equation: $r^2 - r - 12 = 0 \Rightarrow (r+3)(r-4) = 0 \Rightarrow r = -3$ or $r = 4 \Rightarrow y = c_1e^{-3t} + c_2e^{4t}$.
$\boxed{\text{8 pts.}}$

The second d.e. in the given system says $x = y'$, so $x = -3c_1e^{-3t} + 4c_2e^{4t}$. Therefore, the solution of the given system is $\boxed{\boxed{x = -3c_1e^{-3t} + 4c_2e^{4t},\ y = c_1e^{-3t} + c_2e^{4t}}}$ $\boxed{\text{4 pts.}}$